

# LOW REGULARITY LOCAL WELL-POSEDNESS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC INITIAL DATA

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ABSTRACT. The Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition is considered. Local well-posedness for data  $u_0$  in the space  $\widehat{H}_r^s(\mathbb{T})$ , defined by the norms

$$\|u_0\|_{\widehat{H}_r^s(\mathbb{T})} = \|\langle \xi \rangle^s \widehat{u}_0\|_{\ell_{\xi}^{r'}},$$

is shown in the parameter range  $s \geq \frac{1}{2}$ ,  $2 > r > \frac{4}{3}$ . The proof is based on an adaptation of the gauge transform to the periodic setting and an appropriate variant of the Fourier restriction norm method.

## 1. INTRODUCTION AND MAIN RESULT

The Cauchy problem for the derivative nonlinear Schrödinger equation

$$\begin{aligned} i\partial_t u + \partial_x^2 u &= i\partial_x(|u|^2 u) \\ u(0, x) &= u_0(x) \end{aligned} \tag{DNLS}$$

with data  $u_0$  in the classical Sobolev spaces  $H^s(\mathbb{R})$  of functions defined on the real line is known to be locally well-posed for  $s \geq \frac{1}{2}$ . This was shown by Takaoka in [23], where he improved the earlier  $H^1(\mathbb{R})$ -result of Hayashi and Ozawa [14, 13, 15]. His method of proof combines the gauge transform already used by Hayashi and Ozawa with Bourgain's Fourier restriction norm method. A counterexample of Biagioni and Linares [2] shows the optimality of Takaoka's result on the  $H^s(\mathbb{R})$ -scale of data spaces: For  $s < \frac{1}{2}$  the Cauchy problem (DNLS) is ill-posed in the  $C^0$ -uniform sense, although the standard scaling argument suggests local well-posedness for  $s > 0$ . This gap of  $1/2$  derivative between the scaling prediction and Takaoka's result can be closed by leaving the  $H^s(\mathbb{R})$ -scale and considering data in the spaces  $\widehat{H}_r^s(\mathbb{R})$  defined by the norms

$$\|u_0\|_{\widehat{H}_r^s(\mathbb{R})} = \|\langle \xi \rangle^s \widehat{u}_0\|_{L_{\xi}^{r'}}, \quad \langle \xi \rangle = (1 + \xi^2)^{\frac{1}{2}}, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

We remark that these spaces coincide with  $B_{r',k}$  (with weight  $k(\xi) = \langle \xi \rangle^s$ ) introduced by Hörmander, cf. [18], Section 10.1. The idea to consider them as data spaces for nonlinear Schrödinger equations goes back to the work of Cazenave, Vega, and Vilela [4], where corresponding weak norms are used. Yet another alternative class of data spaces has been considered by Vargas and Vega in [24].

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Concerning the (DNLS) equation on the real line, it was shown by the first author in [11], that local well-posedness holds for data in  $\widehat{H}_r^s(\mathbb{R})$ , provided  $s \geq \frac{1}{2}$  and  $2 \geq r > 1$ . This generalization of Takaoka's result almost reaches the critical case, which is  $(s, r) = (\frac{1}{2}, 1)$  in this setting. The proof uses the gauge transform again and an appropriate variant of the Fourier restriction norm method, which was developed in [10]. Furthermore, it relies heavily on certain smoothing properties of the Schrödinger group, expressed in terms of bi- and trilinear estimates for free solutions.

On the other hand it could be shown by the second author in [16], that Takaoka's result concerning the real line can be carried over to the periodic case *with the same lower bound*  $s \geq \frac{1}{2}$  on the Sobolev regularity. This is remarkable, since there is a number of nonlinear Schrödinger and Korteweg - de Vries type equations, which are – due to a lack of smoothing properties – strictly worse behaved in the periodic setting than in the continuous case. To prove the result concerning the one-dimensional torus, the gauge transform had to be adjusted to the periodic case, see Section 2 of [16]. The transformed equation is then treated by the Fourier restriction norm method. Here, the  $L^4$  Strichartz estimate [25, 3] turned out to be a central tool in the derivation of the nonlinear estimates.

Now it is natural to ask for a synthesis of the two last-mentioned results, i. e., to consider the Cauchy problem (DNLS) with  $u_0$  in the following two parameter scale of data spaces.

**Definition 1.1.** Let  $s \in \mathbb{R}$ ,  $1 \leq r \leq \infty$  and  $\frac{1}{r} + \frac{1}{r'} = 1$ . Define  $\widehat{H}_r^s(\mathbb{T})$  as the completion of all trigonometric polynomials with respect to the norm

$$\|f\|_{\widehat{H}_r^s} := \|\widehat{J^s f}\|_{\ell_{\xi}^{r'}}, \quad (1)$$

where  $J^s$  is the Bessel potential operator of order  $-s$  given by  $\widehat{J^s f}(\xi) = \langle \xi \rangle^s \widehat{f}(\xi)$ .

*Remark 1.* In contrast to the non-periodic case the continuous embedding  $\widehat{H}_q^s(\mathbb{T}) \subset \widehat{H}_r^s(\mathbb{T})$  holds true for any  $1 \leq r \leq q \leq \infty$ . Moreover, we have  $H_2^s(\mathbb{T}) = \widehat{H}_2^s(\mathbb{T})$  and more generally  $H_r^s(\mathbb{T}) \subset \widehat{H}_r^s(\mathbb{T})$  for  $1 \leq r \leq 2$  by Hausdorff-Young, where  $H_r^s(\mathbb{T})$  denotes the Bessel potential space of all  $u$  such that  $J^s u \in L^r(\mathbb{T})$ . If  $r = 2$  we will usually omit the index  $r$ .

The main result of this paper is local well-posedness of (DNLS) in these data spaces in the parameter range  $s \geq \frac{1}{2}$  and  $2 > r > \frac{4}{3}$ . More precisely, the following theorem will be shown.

**Theorem 1.2.** Let  $\frac{4}{3} < q \leq r \leq 2$ . For every

$$u_0 \in B_R := \{u_0 \in \widehat{H}_r^{\frac{1}{2}}(\mathbb{T}) \mid \|u_0\|_{\widehat{H}_q^{\frac{1}{2}}} < R\}$$

and  $T \lesssim R^{-2q'}$  there exists a solution  $u \in C([-T, T], \widehat{H}_r^{\frac{1}{2}}(\mathbb{T}))$  of the Cauchy problem (DNLS). This solution is the unique limit of smooth solutions and the map

$$\left( B_R, \|\cdot\|_{\widehat{H}_r^{\frac{1}{2}}} \right) \longrightarrow C([-T, T], \widehat{H}_r^{\frac{1}{2}}(\mathbb{T})) : u_0 \mapsto u$$

is continuous but not locally uniformly continuous. However, on subsets of  $B_R$  with fixed  $L^2$  norm it is locally Lipschitz continuous.

- Remark 2.* (i) The uniqueness statement in the theorem above can be sharpened, see Remark 5.
- (ii) Our methods rely on the  $L^2$  conservation law, but not on the complete integrability of (DNLS), see [19], and also apply to nonlinearities with (say) additional polynomial terms of type  $|u|^k u$
- (iii) Solution always means solution of the corresponding integral equation

$$u(t) = e^{it\partial_x^2} u_0 + i \int_0^t e^{i(t-t')\partial_x^2} \partial_x (|u|^2 u)(t') dt', \quad t \in (-T, T).$$

- (iv) In view of the counterexamples in [16], Theorem 5.3, and [17], Theorem 3.1.5, which are essentially of the same kind as the one already given in [23], Proposition 3.3, we cannot expect any positive result for  $s < \frac{1}{2}$ . Observe that the examples concerning the periodic case are monochromatic waves and so do not distinguish between an  $\ell_\xi^2$ - and an  $\ell_\xi^{r'}$ -norm. Concerning the second parameter  $r$ , we must leave open the question, whether or not there is local well-posedness for  $r \leq \frac{4}{3}$ . Nonetheless, we will show below that our result is optimal within the framework we use.

Before we turn to details, let us point out, that in the periodic case almost nothing is known about Cauchy problems with data in the  $\widehat{H}_r^s(\mathbb{T})$  spaces. The only result we are aware of is due to Christ [5, 6], who considers the following modification of the cubic nonlinear Schrödinger equation on the one-dimensional torus

$$i\partial_t u + \partial_x^2 u = \left( |u|^2 - 2 \int_0^{2\pi} |u|^2 dx \right) u, \quad (\text{NLS}^*)$$

with initial condition  $u(0) = u_0 \in \widehat{H}_r^s(\mathbb{T})$ . He shows that for  $s \geq 0$  and  $r > 1$  the solution map

$$S : H^\sigma(\mathbb{T}) \longrightarrow C([0, \infty), H^\sigma(\mathbb{T})) \cap C^1([0, \infty), H^{\sigma-2}(\mathbb{T}))$$

( $\sigma$  sufficiently large) “extends by continuity to a uniformly continuous mapping from the ball centered at 0 of [arbitrary] radius  $R$  in  $\widehat{H}_r^s(\mathbb{T})$  to  $C([0, \tau], \widehat{H}_r^s(\mathbb{T}))$ ”, where  $\tau$  depends on  $R$ , see Theorem 1.1 in [5].<sup>1</sup> This result is shown by a *new method of solution*, which is developed in [5], a summary of this method is given in Section 1.5 of that paper. The positive result in [5] is supplemented in [6] by a statement of non-uniqueness: For  $2 > r > 1$  there exists a non-vanishing weak solution  $u \in C([0, 1], \widehat{H}_r^0(\mathbb{T}))$  of (NLS\*) with initial value  $u_0 \equiv 0$ , see Theorem 2.3 in [6].<sup>2</sup>

Using the function spaces  $X_r^{s,b}$ , defined by the norms

$$\|u\|_{X_r^{s,b}} = \|\langle \xi \rangle^s \langle \tau + \xi^2 \rangle^b \widehat{u}\|_{\ell_\xi^{r'} L_\tau^{r'}},$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$  (cf. [10], Section 2), we can actually show local *well-posedness* of the initial value problem associated to (NLS\*) with data  $u_0 \in \widehat{H}_r^0(\mathbb{T})$ ,  $2 > r > 1$ , thus giving an alternative proof (based on the contraction mapping principle) of Christ’s result from [5]. The argument also provides uniqueness of the solution in the restriction norm space based on  $X_r^{0,b}$ , see (2.37) and (2.38) in [10]. It was the

<sup>1</sup>In [5] the data spaces are denoted as  $\mathcal{H}^{s,p}(\mathbb{T})$ , which corresponds to  $\widehat{H}_{p'}^s(\mathbb{T})$  in our terms.

<sup>2</sup>A precise definition of a weak solution is given in [6], Section 2.1.

starting point for our investigations concerning the (DNLS) equation and exhibits already some of the main arguments, so let us sketch this proof:

We define the trilinear Operator  $C_1$  by its partial Fourier transform (in the space variable only)

$$C_1(\widehat{u_1, u_2, u_3})(\xi) = (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \widehat{u_1}(\xi_1) \widehat{u_2}(\xi_2) \widehat{u_3}(\xi_3),$$

so that the (partial) Fourier transform of the nonlinearity in (NLS\*) becomes

$$C_1(\widehat{u, u, u})(\xi) = (2\pi)^{-1} \widehat{u^2}(\xi) \widehat{u}(-\xi).$$

By Theorem 2.3 from [10] it is sufficient to estimate the latter appropriately in  $X_r^{0,b}$ -norms. Here, the second contribution turns out to be harmless, cf. the end of the proof of Theorem 2.4 below. So matters essentially reduce to show the following estimate:

**Proposition 1.3.** *Let  $r > 1$ ,  $\varepsilon > 0$  and  $b > \frac{1}{r}$ . Then*

$$\|C_1(u_1, u_2, u_3)\|_{X_r^{0,-\varepsilon}} \lesssim \prod_{i=1}^3 \|u_i\|_{X_r^{0,b}}.$$

*Proof.* Choosing  $f_i \in \ell_{\xi}^{r'} L_{\tau}^{r'}$  such that  $\|f_i\|_{\ell_{\xi}^{r'} L_{\tau}^{r'}} = \|u_i\|_{X_r^{0,b}}$  the above estimate can be rewritten as

$$\left\| \langle \sigma_0 \rangle^{-\varepsilon} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int_{\tau = \tau_1 + \tau_2 + \tau_3} \prod_{i=1}^3 \frac{f_i(\xi_i, \tau_i)}{\langle \sigma_i \rangle^b} d\tau_1 d\tau_2 \right\|_{\ell_{\xi}^{r'} L_{\tau}^{r'}} \lesssim \prod_{i=1}^3 \|f_i\|_{\ell_{\xi}^{r'} L_{\tau}^{r'}}, \quad (2)$$

where  $\sigma_0 = \tau + \xi^2$ ,  $\sigma_i = \tau_i + \xi_i^2$  ( $i = 1, 2$ ) and  $\sigma_3 = \tau_3 - \xi_3^2$ . By Hölder's inequality and Fubini's theorem (2) can be deduced from

$$\sup_{\xi, \tau} \langle \sigma_0 \rangle^{-r\varepsilon} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int_{\tau = \tau_1 + \tau_2 + \tau_3} \prod_{i=1}^3 \langle \sigma_i \rangle^{-rb} d\tau_1 d\tau_2 < \infty. \quad (3)$$

Using the resonance relation

$$2|\xi_1 \xi_2 + \xi \xi_3| = 2|\xi - \xi_1| |\xi - \xi_2| \leq \sum_{i=0}^3 \langle \sigma_i \rangle \leq \prod_{i=0}^3 \langle \sigma_i \rangle \quad (4)$$

the left hand side of (3) is bounded by

$$\begin{aligned} & \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \int_{\tau = \tau_1 + \tau_2 + \tau_3} d\tau_1 d\tau_2 \prod_{i=1}^3 \langle \sigma_i \rangle^{-1-} \\ & \lesssim \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \langle \tau + \xi^2 - 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-}, \end{aligned}$$

where in the last step we have used Lemma 4.1 twice. Setting  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ ,  $n_i = \xi - \xi_i$  for  $i = 1, 2$  and  $r = n_1 n_2$  the last sum can be rewritten as

$$\sum_{r \in \mathbb{Z}^*} \langle \tau + \xi^2 - 2r \rangle^{-1-} \langle r \rangle^{0-} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ r = n_1 n_2}} 1$$

which is bounded by a constant independent of  $\xi$  and  $\tau$ , since the number of divisors of  $r \in \mathbb{N}$  can be estimated by  $c_\varepsilon r^\varepsilon$  for any positive  $\varepsilon$ .  $\square$

Three aspects of the preceding are worth to be emphasized in view of our investigations here.

**Necessity of cancellations and correction terms:** For  $r < 2$  the above argument breaks down completely without the restrictions  $\xi \neq \xi_1$  and  $\xi \neq \xi_2$  in the sum over the Fourier coefficients. As was pointed out already by Christ, this cancellation comes from the correction term  $2\int_0^{2\pi} |u|^2 dx u$  subtracted in the nonlinearity, and for any other coefficient in front of this term one cannot obtain continuous dependence (see [5], last sentence of Section 1.3 and the remark before (2.6)). A very similar cancellation turns out to be fundamental in our analysis of the (DNLS) equation, but here *the corresponding correction term comes from the gauge transform* in its periodic variant, which is discussed in Section 6, Remark 4 below. In fact, the main contribution to the cubic part of the transformed equation is given by  $T^*(u, u, u)$ , where

$$T^*(\widehat{u_1, u_2, u_3})(\xi) = (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) i \xi_3 \widehat{u}_3(\xi_3).$$

Again, our argument would not work without the restrictions  $\xi \neq \xi_1, \xi \neq \xi_2$ .

**Modification of the norms:** If we try to estimate the term  $T^*(u_1, u_2, u_3)$  in an  $X_r^{s,b}$ -norm in a similar manner as in the proof of Proposition 1.3, we have to get control over a whole derivative, that is, on Fourier side, over the factor  $\xi_3$ . The complete absence of smoothing effects (gaining derivatives) in the periodic case forces us to get this control from the resonance relation (4) only, which is the same for (DNLS) as for (NLS\*). This means, that we have to choose the  $b$ -parameters equal to  $-\frac{1}{2}$  on the left and to  $+\frac{1}{2}$  on the right hand side of the estimate. Now, the necessity to cancel the  $\xi_3$ -factor and the resonance relation lead to the consideration of eight cases – some of them being symmetric – depending on which of the  $\sigma$ 's is maximal and on whether or not  $|\xi \xi_3| \lesssim |\xi_1 \xi_2|$ , see the table in the proof of Theorem 2.4 below. Picking out the (relatively harmless) subcase, where  $|\xi \xi_3| \lesssim |\xi_1 \xi_2|$  and  $\sigma_0$  is maximal, so that  $\prod_{i=1}^3 \langle \sigma_i \rangle^{\frac{1}{6}} \leq \langle \sigma_0 \rangle^{\frac{1}{2}}$ , we are in the situation of the above proof, with a half derivative on each factor (as desired) but with a  $b$ -parameter on the right of at most  $\frac{2}{3}$ , which means that we end up with the non-optimal restriction  $r > \frac{3}{2}$ . This leads us to *introduce a fourth parameter  $p$*  in the  $X_r^{s,b}$ -norms, which is the Hölder exponent concerning the  $\tau$ -integration and may differ from  $r$ , see Definition 2.1 below. In our application here we choose  $p = 2$ , thus going back to some extent to the meanwhile classical  $X^{s,b}$ -spaces.

**Number of divisor estimate:** The number of divisor argument at the end of the proof of Proposition 1.3 has been used already in Christ's work and can be seen as a substitute for Bourgain's  $L^6$  Strichartz estimate for the periodic case, which itself was shown by the aid of this argument, see Proposition 2.36 in [3]. We will need a refined version thereof, which is shown by elementary geometric considerations in Section 3. Here, we use arguments similar to those of De Silva, Pavlovic, Staffilani, and Tzirakis [8], Section 4.

Concerning the organization of the paper the following should be added: In Section 2 we introduce the relevant function spaces and state all the nonlinear

estimates needed as well as a sharpness result. The crucial trilinear estimates and a counterexample are derived in Section 4, which is very much in the spirit of [20]. Section 5 deals with the quintilinear estimate. In both cases we have made some effort to extract the correct lifespan from the nonlinear estimates and to obtain persistence of higher regularity. By this we mean that the lifespan of a solution with  $\widehat{H}_r^{\frac{1}{2}}(\mathbb{T})$ -data only depends on the smaller  $\widehat{H}_q^{\frac{1}{2}}(\mathbb{T})$ -norm of the initial value, where  $2 \geq r > q > \frac{4}{3}$ .<sup>3</sup> Finally, in Section 7, the contraction mapping principle is invoked to prove local well-posedness for the transformed equation (49), see Theorem 7.2. Our main result, Theorem 1.2, is then a consequence of Lemma 6.4 on the gauge transform.

We close this section by fixing some notational conventions.

- The Fourier transform with respect to the space variable (periodic)

$$\mathcal{F}_x f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-ix\xi} dx \quad (f \in L^1(\mathbb{T}))$$

- The Fourier transform with respect to the time variable (nonperiodic)

$$\mathcal{F}_t f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\tau} dt \quad (f \in L^1(\mathbb{R}))$$

- The Fourier transform with respect to time and space variables  $\mathcal{F} = \mathcal{F}_t \mathcal{F}_x$
- For the mean value integral we write

$$\oint_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \quad (f \in L^1(\mathbb{T}))$$

- Let  $a \in \mathbb{R}$ . The expressions  $a \pm$  denote numbers  $a \pm \varepsilon$  for an arbitrarily small  $\varepsilon > 0$
- For a given set of parameters (typically a subset of  $\varepsilon, \delta, \nu, p, q, r, s$ ) the statement  $A \lesssim B$  means that there exists a constant  $C > 0$  which depends only on these parameters such that  $A \leq CB$ . This is equivalent to  $B \gtrsim A$ . We may write  $A \ll B$  if it is possible to choose  $0 < C < \frac{1}{4}$ .
- For all parameters  $1 \leq p \leq \infty$  the number  $1 \leq p' \leq \infty$  is defined to be the dual parameter satisfying  $\frac{1}{p} + \frac{1}{p'} = 1$ .

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## 2. FUNCTION SPACES AND MAIN ESTIMATES

Let  $\mathcal{S}(\mathbb{R} \times \mathbb{T})$  be the linear space of all  $C^\infty$ -functions  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  such that

$$f(t, x) = f(t, x + 2\pi), \quad \sup_{(t, x) \in \mathbb{R}^2} |t^\alpha \partial_t^\beta \partial_x^\gamma f(t, x)| < \infty, \quad \alpha, \beta, \gamma \in \mathbb{N}_0$$

**Definition 2.1.** Let  $s, b \in \mathbb{R}$ ,  $1 \leq r, p \leq \infty$  and  $\frac{1}{r} + \frac{1}{r'} = 1 = \frac{1}{p} + \frac{1}{p'}$ . Define the space  $X_{r,p}^{s,b}$  as the completion of  $\mathcal{S}(\mathbb{R} \times \mathbb{T})$  with respect to the norm

$$\|u\|_{X_{r,p}^{s,b}} = \|\langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s \mathcal{F}u\|_{\ell_{\xi}^{r'} L_{\tau}^{p'}} \quad (5)$$

<sup>3</sup>We refer to [22], Theorem 2, part V, for the corresponding notion, if data in the  $H^s$ -scale are considered. In the  $H^s$ -case this property usually is a simple consequence of the convolution constraint - see again [22], Remark 2 below Theorem 3. We cannot see that a similar argument should work in our setting.

In the case where  $r = p = 2$  we write  $X_{r,p}^{s,b} = X^{s,b}$  as usual.

**Lemma 2.2.** *Let  $s, b_1, b_2 \in \mathbb{R}$ ,  $1 \leq r \leq \infty$  and  $b_1 > b_2 + \frac{1}{2}$ . The following embeddings are continuous:*

$$X_{r,2}^{s,b_1} \subset X_{r,\infty}^{s,b_2} \quad (6)$$

$$X_{r,\infty}^{s,0} \subset C(\mathbb{R}, \widehat{H}_r^s(\mathbb{T})) \quad (7)$$

*Proof.* The first embedding is proved by the Cauchy-Schwarz inequality with respect to the  $L_\tau^1$  norm. The second embedding follows from  $L^\infty(\mathbb{R}) \subset \mathcal{F}_t^{-1}L^1(\mathbb{R})$ .  $\square$

**Definition 2.3.** Let  $s \in \mathbb{R}$  and  $1 \leq r \leq \infty$ . We define

$$Z_r^s := X_{r,2}^{s,\frac{1}{2}} \cap X_{r,\infty}^{s,0}$$

and for  $0 < T \leq 1$  the restriction space  $Z_r^s(T)$  of all  $v = w|_{[-T,T]}$  for some  $w \in Z_r^s$  with norm

$$\|v\|_{Z_r^{\frac{1}{2}}(T)} := \inf\{\|w\|_{Z_r^{\frac{1}{2}}} \mid w \in Z_r^{\frac{1}{2}} : w|_{[-T,T]} = v\}$$

A main ingredient for the proof of Theorem 1.2 is an estimate on the trilinear operator (suppressing the  $t$  dependence)

$$\begin{aligned} T(\widehat{u_1, u_2, u_3})(\xi) &= (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) i \xi_3 \widehat{u}_3(\xi_3) \\ &\quad + (2\pi)^{-1} \widehat{u}_1(\xi) \widehat{u}_2(\xi) i \xi \widehat{u}_3(-\xi) \end{aligned} \quad (8)$$

**Theorem 2.4.** *Let  $\frac{4}{3} < q \leq r \leq 2$  and  $0 \leq \delta < \frac{1}{q}$ . Then,*

$$\|T(u_1, u_2, u_3)\|_{X_{r,2}^{\frac{1}{2}, -\frac{1}{2}}} \lesssim T^\delta \|u_1\|_{X_{q,2}^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{q,2}^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{r,2}^{\frac{1}{2}, \frac{1}{2}}} \quad (9)$$

if  $\text{supp}(u_i) \subset \{(t, x) \mid |t| \leq T\}$ ,  $0 < T \leq 1$ .

Additionally, we will need the following estimate on  $T(u_1, u_2, u_3)$ .

**Theorem 2.5.** *Let  $\frac{4}{3} < q \leq r \leq 2$  and  $0 \leq \delta < \frac{1}{q}$ . Then,*

$$\|T(u_1, u_2, u_3)\|_{X_{r,\infty}^{\frac{1}{2}, -1}} \lesssim T^\delta \|u_1\|_{X_{q,2}^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{q,2}^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{r,2}^{\frac{1}{2}, \frac{1}{2}}} \quad (10)$$

if  $\text{supp}(u_i) \subset \{(t, x) \mid |t| \leq T\}$ ,  $0 < T \leq 1$ .

The above estimates are sharp with respect to the lower threshold on  $r$  within the full scale of spaces  $X_{r,p}^{\frac{1}{2}, \frac{1}{2}}$ . Note that in particular the estimates fail to hold in the endpoint case  $r = \frac{4}{3}$ .

*Remark 3.* For all  $b \leq 0$ ,  $1 \leq r \leq \frac{4}{3}$  and  $1 \leq p, q \leq \infty$  the estimate

$$\|T(u_1, u_2, u_3)\|_{X_{r,p}^{\frac{1}{2}, b}} \lesssim \prod_{i=1}^3 \|u_i\|_{X_{r,q}^{\frac{1}{2}, \frac{1}{2}}} \quad (11)$$

is false.

We also consider the quintilinear expression defined as

$$Q(\widehat{u_1, \dots, u_5})(\xi) = (2\pi)^{-2} \sum_{**} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \widehat{u}_3(\xi_3) \widehat{u}_4(\xi_4) \widehat{u}_5(\xi_5) \quad (12)$$

where we suppressed the  $t$  dependence and  $**$  is shorthand for summation over the subset of  $\mathbb{Z}^5$  given by the restrictions

$$\xi = \xi_1 + \dots + \xi_5 ; \xi_1 + \dots + \xi_4 \neq 0 ; \xi_1 + \xi_2 \neq 0 ; \xi_3 + \xi_4 \neq 0$$

**Theorem 2.6.** *Let  $\frac{4}{3} < q \leq r \leq 2$  and  $b > \frac{1}{6} + \frac{1}{3q}$ . Then,*

$$\|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X_{r,2}^{\frac{1}{2},-b}} \lesssim \sum_{k=1}^5 \|u_k\|_{X_{r,2}^{\frac{1}{2},b}} \prod_{\substack{1 \leq i \leq 5 \\ i \neq k}} \|u_i\|_{X_{q,2}^{\frac{1}{2},b}} \quad (13)$$

*Additionally assume that for  $0 < T \leq 1$  we have  $\text{supp}(u_i) \subset \{(t, x) \mid |t| \leq T\}$  and  $0 \leq \delta < \frac{2}{q}$ . Then,*

$$\|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X_{r,2}^{\frac{1}{2},-\frac{1}{2}} \cap X_{r,\infty}^{\frac{1}{2},-1}} \lesssim T^\delta \sum_{k=1}^5 \|u_k\|_{X_{r,2}^{\frac{1}{2},\frac{1}{2}}} \prod_{\substack{1 \leq i \leq 5 \\ i \neq k}} \|u_i\|_{X_{q,2}^{\frac{1}{2},\frac{1}{2}}} \quad (14)$$

and

$$\|Q(u_1, u_2, u_3, u_4, u_5)\|_{X_{r,2}^{\frac{1}{2},-\frac{1}{2}} \cap X_{r,\infty}^{\frac{1}{2},-1}} \lesssim T^\delta \sum_{k=1}^5 \|u_k\|_{X_{r,2}^{\frac{1}{2},\frac{1}{2}}} \prod_{\substack{1 \leq i \leq 5 \\ i \neq k}} \|u_i\|_{X_{q,2}^{\frac{1}{2},\frac{1}{2}}} \quad (15)$$

### 3. NUMBER OF DIVISOR ESTIMATES AND CONSEQUENCES

The next lemma contains estimates on the number of divisors of a given natural number  $r$ . Part (i) is well-known (see Hardy-Wright [12], Theorem 315). The approach used to prove Part (ii) of Lemma 3.1 is motivated by [8], Lemma 4.4.

**Lemma 3.1.** (i) *Let  $\varepsilon > 0$ . There exists  $c_\varepsilon > 0$ , such that for all  $r \in \mathbb{N}$*

$$\#\{(n_1, n_2) \in \mathbb{N}^2 \mid n_1 n_2 = r\} \leq c_\varepsilon r^\varepsilon \quad (16)$$

(ii) *For all  $r \in \mathbb{N}$*

$$\#\left\{(n_1, n_2) \in \mathbb{N}^2 \mid n_1 n_2 = r, \quad 3|n_1 - n_2| \leq r^{\frac{1}{6}}\right\} \leq 2 \quad (17)$$

*Proof of Part (ii).* Let  $r \in \mathbb{N}$ . Assume that there are three lattice points contained in the above set. Then, these points form a triangle of area  $\mu \geq \frac{1}{2}$ . This triangle is located

(i) in the strip

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid \sqrt{r} - \delta \leq x_1 \leq \sqrt{r} + \delta\}$$

where  $\delta = \frac{1}{3}r^{\frac{1}{6}}$ , because  $|n_1 - \sqrt{r}| \leq |n_1 - n_2| \leq \delta$ .

(ii) below the line  $L = \overline{P_1 P_2}$  which connects the points

$$P_1 = \left(\sqrt{r} - \delta, \frac{r}{\sqrt{r} - \delta}\right) \quad , \quad P_2 = \left(\sqrt{r} + \delta, \frac{r}{\sqrt{r} + \delta}\right)$$

(iii) above the hyperbola

$$H = \{(x_1, x_2) \in (0, \infty)^2 \mid x_1 x_2 = r\}$$

because the function  $x \mapsto \frac{r}{x}$  is convex for  $x > 0$ .



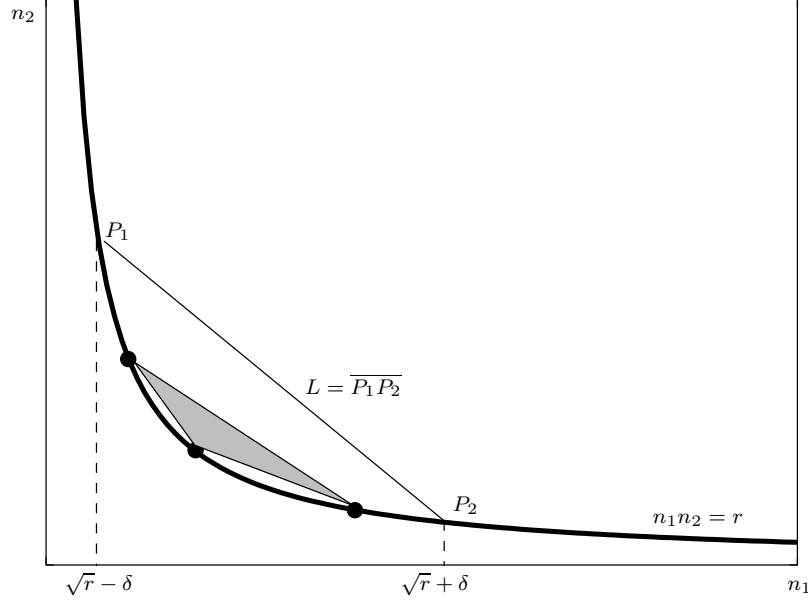


FIGURE 1. The notional triangle from the proof of Lemma 3.1

Now, the area  $\mu$  of the triangle is bounded from above by the area of the region in the strip  $S$  below  $L$  and above  $H$ , hence

$$\begin{aligned}
 \mu &\leq r\delta \left( \frac{1}{\sqrt{r}-\delta} + \frac{1}{\sqrt{r}+\delta} \right) - \int_{\sqrt{r}-\delta}^{\sqrt{r}+\delta} \frac{r}{x} dx \\
 &= r \left( \frac{2\sqrt{r}\delta}{r-\delta^2} - \ln \left( 1 + \frac{2\delta}{\sqrt{r}-\delta} \right) \right) \\
 &\leq r \left( \frac{2\sqrt{r}\delta}{r-\delta^2} - \frac{2\delta}{\sqrt{r}-\delta} + \frac{2\delta^2}{(\sqrt{r}-\delta)^2} \right) = \frac{4r\delta^3}{(\sqrt{r}-\delta)^2(\sqrt{r}+\delta)} \\
 &\leq \frac{4\sqrt{r}\delta^3}{(\sqrt{r}-\delta)^2} \leq \frac{4}{27} \frac{r}{(\sqrt{r}-\delta)^2} \leq \frac{1}{3}
 \end{aligned}$$

which contradicts  $\mu \geq \frac{1}{2}$ . □

Now, we use Lemma 3.1 to prove

**Corollary 3.2.** Fix  $\varepsilon > 0$ .

(i) There exists  $C_\varepsilon > 0$  such that for all  $\xi \in \mathbb{Z}$  and  $a \in \mathbb{R}$

$$\sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{-\varepsilon} \langle \xi - \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \leq C_\varepsilon \quad (18)$$

(ii) There exists  $C_\varepsilon > 0$  such that for all  $\xi_1 \in \mathbb{Z}$  and  $a \in \mathbb{R}$

$$\sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{-\varepsilon} \langle \xi - \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \leq C_\varepsilon \quad (19)$$

(iii) There exists  $C_\varepsilon > 0$  such that for all  $\xi \in \mathbb{Z}$  and  $a \in \mathbb{R}$

$$\sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-\varepsilon} \langle \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \leq C_\varepsilon \quad (20)$$

(iv) There exists  $C_\varepsilon > 0$  such that for all  $\xi_1 \in \mathbb{Z}$  and  $a \in \mathbb{R}$

$$\langle \xi_1 \rangle^{-\varepsilon} \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \leq C_\varepsilon \quad (21)$$

*Proof.* The first and second part follow from the standard number of divisors estimate (16) as follows: By the change of variables  $n_1 = \xi - \xi_1$ ,  $n_2 = \xi - \xi_2$  the sums in (18) and (19) are equal to

$$\sum_{n_1, n_2 \in \mathbb{Z}^*} \langle n_1 \rangle^{-\varepsilon} \langle n_2 \rangle^{-\varepsilon} \langle a + 2n_1 n_2 \rangle^{-1-\varepsilon}$$

This can be written as

$$\begin{aligned} & \sum_{r \in \mathbb{Z}^*} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n_1 n_2 = r}} \langle n_1 \rangle^{-\varepsilon} \langle n_2 \rangle^{-\varepsilon} \langle a + 2n_1 n_2 \rangle^{-1-\varepsilon} \\ & \leq \sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon} \langle r \rangle^{-2\varepsilon} \# \{ (n_1, n_2) \in (\mathbb{Z}^*)^2 \mid n_1 n_2 = r \} \\ & \leq c_\varepsilon \sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon} \end{aligned}$$

for some  $c_\varepsilon > 1$ . We write  $a = 2b + \delta$ ,  $b \in \mathbb{Z}$ ,  $\delta \in [0, 2)$  and

$$\sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon} \leq \sum_{r \in \mathbb{Z}} \langle r + \delta \rangle^{-1-\varepsilon} \leq 3 + 2 \sum_{r \in \mathbb{N}} \langle r \rangle^{-1-\varepsilon} =: s_\varepsilon$$

Now, the estimates (18) and (19) hold with  $C_\varepsilon := s_\varepsilon c_\varepsilon$ . In order to show formula (20) of the third part we use the same change of variables as above and obtain

$$\sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n_1 n_2 = r}} \langle \xi - n_1 \rangle^{-\varepsilon} \langle \xi - n_2 \rangle^{-\varepsilon}$$

Let  $M(r) = \{ (n_1, n_2) \in (\mathbb{Z}^*)^2 \mid n_1 n_2 = r \}$ . Now, we split the inner sum into two parts. Let

$$M_1(r) = \left\{ (n_1, n_2) \in M(r) \mid 6|\xi - n_1| \geq |r|^{\frac{1}{6}} \text{ or } 6|\xi - n_2| \geq |r|^{\frac{1}{6}} \right\}$$

and

$$M_2(r) = \left\{ (n_1, n_2) \in M(r) \mid 6|\xi - n_1| \leq |r|^{\frac{1}{6}} \text{ and } 6|\xi - n_2| \leq |r|^{\frac{1}{6}} \right\}$$

Obviously we have  $M(r) = M_1(r) \cup M_2(r)$ . By Part (i) of Lemma 3.1 there exists  $c_\varepsilon > 1$  such that

$$\#M_1(r) \leq 2 \# \{ (n_1, n_2) \in \mathbb{N}^2 \mid n_1 n_2 = r \} \leq c_\varepsilon |r|^{\frac{\varepsilon}{6}}$$

and it follows

$$\sum_{(n_1, n_2) \in M_1(r)} \langle \xi - n_1 \rangle^{-\varepsilon} \langle \xi - n_2 \rangle^{-\varepsilon} \leq 6^\varepsilon |r|^{-\frac{\varepsilon}{6}} \#M_1(r) \leq 6^\varepsilon c_\varepsilon$$

For  $(n_1, n_2) \in M_2(r)$  it holds that  $3|n_1 - n_2| \leq |r|^{\frac{1}{6}}$ . An application of Part (ii) of Lemma 3.1 shows

$$\sum_{(n_1, n_2) \in M_2(r)} \langle \xi - n_1 \rangle^{-\varepsilon} \langle \xi - n_2 \rangle^{-\varepsilon} \leq \#M_2(r) \leq 4$$

Therefore, we see that

$$\sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-\varepsilon} \langle \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \leq (6^\varepsilon c_\varepsilon + 4) \sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon}$$

and the third part is proved with constant  $C_\varepsilon = s_\varepsilon(6^\varepsilon c_\varepsilon + 4)$ . Concerning the fourth part we proceed similarly. After changing variables

$$\begin{aligned} & \langle \xi_1 \rangle^{-\varepsilon} \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_2 \rangle^{-\varepsilon} \langle a + 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1-\varepsilon} \\ &= \langle \xi_1 \rangle^{-\varepsilon} \sum_{r \in \mathbb{Z}^*} \langle a + 2r \rangle^{-1-\varepsilon} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n_1 n_2 = r}} \langle \xi_1 + n_1 - n_2 \rangle^{-\varepsilon} \end{aligned}$$

we consider two subregions of summation. In the case where  $|r|^{\frac{1}{6}} \leq 6|\xi_1|$  or  $|r|^{\frac{1}{6}} \leq 6|\xi_1 + n_1 - n_2|$  we apply estimate (16) from the first part of Lemma 3.1, while in the remaining case it holds  $3|n_1 - n_2| \leq |r|^{\frac{1}{6}}$  and we utilize estimate (17) from the second part of Lemma 3.1.  $\square$

#### 4. THE PROOF OF THE TRILINEAR ESTIMATES

In this section we prove Theorem 2.4 and Theorem 2.5. We will frequently use the following well-known (see e.g. [9], Lemma 4.2) tool:

**Lemma 4.1.** *Let  $0 \leq \alpha \leq \beta$  such that  $\alpha + \beta > 1$  and  $\varepsilon > 0$ . Then,*

$$\int_{\mathbb{R}} \langle s - a \rangle^{-\alpha} \langle s - b \rangle^{-\beta} ds \lesssim \langle a - b \rangle^{-\gamma}, \quad \gamma = \begin{cases} \alpha + \beta - 1 & , \beta < 1 \\ \alpha - \varepsilon & , \beta = 1 \\ \alpha & , \beta > 1 \end{cases}$$

We write  $T = T^* + T^{**}$ , where

$$\begin{aligned} T^*(\widehat{u_1, u_2, u_3})(\xi) &= (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \widehat{u_1}(\xi_1) \widehat{u_2}(\xi_2) i \xi_3 \widehat{u_3}(\xi_3) \\ T^{**}(\widehat{u_1, u_2, u_3})(\xi) &= (2\pi)^{-1} \widehat{u_1}(\xi) \widehat{u_2}(\xi) i \xi \widehat{u_3}(-\xi) \end{aligned}$$

*Proof of Theorem 2.4.* To fix notation, let  $\sigma_0 = \tau + \xi^2$ ,  $\sigma_j = \tau_j + \xi_j^2$ ,  $j = 1, 2$  and  $\sigma_3 = \tau_3 - \xi_3^2$ . Throughout the proof the quantities  $\xi_3, \tau_3$  are defined as  $\xi_3 = \xi - \xi_1 - \xi_2$  and  $\tau_3 = \tau - \tau_1 - \tau_2$ , respectively. Let us denote  $\mu = (\tau, \xi)$ ,  $\mu_i = (\tau_i, \xi_i)$ ,  $i = 1, 2, 3$  for brevity. By the definition of the norms we may assume that  $\widehat{u_j} \geq 0$ . Then,

$$\|T(u_1, u_2, u_3)\|_{X_{r,2}^{\frac{1}{2}, -\frac{1}{2}}} \leq \|T^*(u_1, u_2, u_3)\|_{X_{r,2}^{\frac{1}{2}, -\frac{1}{2}}} + \|T^{**}(u_1, u_2, u_3)\|_{X_{r,2}^{\frac{1}{2}, -\frac{1}{2}}}$$

and we consider the contribution from  $T^*$  first: Let  $m$  be given by

$$m(\mu, \mu_1, \mu_2) = \frac{\langle \xi \rangle^{\frac{1}{2}} i \xi_3}{\prod_{j=1}^3 \langle \xi_j \rangle^{\frac{1}{2}} \prod_{j=0}^3 \langle \sigma_j \rangle^{\frac{1}{2}}}$$

Estimate (9) for the  $T^*$  contribution is equivalent to

$$\left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int m(\mu, \mu_1, \mu_2) f_1(\mu_1) f_2(\mu_2) f_3(\mu_3) d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^2} \quad (22)$$

$$\lesssim T^\delta \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

where we may assume  $f_3(\tau_3, 0) = 0$ . The resonance relation

$$\sigma_0 - \sigma_1 - \sigma_2 - \sigma_3 = 2(\xi - \xi_1)(\xi - \xi_2) = 2(\xi_1 \xi_2 + \xi \xi_3) \quad (23)$$

holds true, cp. [23, 11, 16]. Let us first consider the subregion where  $\langle \xi_1 \rangle \langle \xi_2 \rangle \ll \langle \xi \rangle \langle \xi_3 \rangle$ . Then,

$$\langle \xi \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}} \lesssim \sum_{k=0}^3 \langle \sigma_k \rangle^{\frac{1}{2}} \quad (24)$$

and in this subregion we control  $|m|$  by the sum of all

$$m_{k,1}(\mu, \mu_1, \mu_2) = \frac{1}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \prod_{j=0, j \neq k}^3 \langle \sigma_j \rangle^{\frac{1}{2}}}$$

for  $k = 0, \dots, 3$ . Secondly, in the subregion where  $\langle \xi \rangle \langle \xi_3 \rangle \lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle$  (note that  $\xi_1 \neq \xi, \xi_2 \neq \xi$  within the domain of summation) it holds

$$\langle \xi - \xi_1 \rangle^{\frac{1}{2}} \langle \xi - \xi_2 \rangle^{\frac{1}{2}} \lesssim \sum_{k=0}^3 \langle \sigma_k \rangle^{\frac{1}{2}} \quad (25)$$

and in this subregion we control  $|m|$  by the sum of all

$$m_{k,2}(\mu, \mu_1, \mu_2) = \frac{1}{\langle \xi - \xi_1 \rangle^{\frac{1}{2}} \langle \xi - \xi_2 \rangle^{\frac{1}{2}} \prod_{j=0, j \neq k}^3 \langle \sigma_j \rangle^{\frac{1}{2}}}$$

for  $k = 0, \dots, 3$ . According to these multipliers we subdivide the proof into the following cases (with a preview of the lower bound on  $q$  for each subcase obtained by our arguments below):

	$\sigma_0 = \max$	$\sigma_1 = \max$	$\sigma_2 = \max$	$\sigma_3 = \max$
$\langle \xi_1 \rangle \langle \xi_2 \rangle \ll \langle \xi \rangle \langle \xi_3 \rangle$	Case 0.1: $q > 1$	Case 1.1: $q > 4/3$	Case 2.1: $q > 4/3$	Case 3.1: $q > 4/3$
$\langle \xi \rangle \langle \xi_3 \rangle \lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle$	Case 0.2: $q > 1$	Case 1.2: $q > 4/3$	Case 2.2: $q > 4/3$	Case 3.2: $q > 4/3$

For technical reasons, we will prove the slightly stronger estimates

$$\left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int m_{k,j,\nu}(\mu, \mu_1, \mu_2) f_1(\mu_1) f_2(\mu_2) f_3(\mu_3) d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^2} \quad (26)$$

$$\lesssim \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

for any  $0 \leq \nu < \frac{1}{3q'}$ ,  $k = 0, \dots, 3$  and  $j = 1, 2$ , where

$$m_{k,1,\nu}(\mu, \mu_1, \mu_2) = \frac{1}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \prod_{j=0, j \neq k}^3 \langle \sigma_j \rangle^{\frac{1}{2}-\nu}}$$

$$m_{k,2,\nu}(\mu, \mu_1, \mu_2) = \frac{1}{\langle \xi - \xi_1 \rangle^{\frac{1}{2}} \langle \xi - \xi_2 \rangle^{\frac{1}{2}} \prod_{j=0, j \neq k}^3 \langle \sigma_j \rangle^{\frac{1}{2}-\nu}}$$

for  $k = 0, \dots, 3$ . Clearly, (26) implies (22) with  $\delta = 3\nu$  because

$$\| \langle \sigma_j \rangle^{-\nu} f_j \|_{\ell_{\xi}^{p'} L_{\tau}^2} \lesssim T^{\nu} \| f_j \|_{\ell_{\xi}^{p'} L_{\tau}^2}, \quad 1 \leq p \leq \infty \quad (27)$$

*Case 0.1:* We consider the contribution

$$t_{0,1} := \left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{f_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}-\nu}} \frac{f_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}-\nu}} \frac{f_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2}-\nu}} d\tau_1 d\tau_2 \right\|_{\ell_{\xi}^{r'} L_{\tau}^2}$$

$$\lesssim \left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} I(\mu, \mu_1, \mu_2) \left( \int f_1^2 f_2^2 f_3^2 d\tau_1 d\tau_2 \right)^{\frac{1}{2}} \right\|_{\ell_{\xi}^{r'} L_{\tau}^2}$$

where

$$I(\mu, \mu_1, \mu_2) := \left( \int \frac{d\tau_1 d\tau_2}{(\langle \sigma_1 \rangle \langle \sigma_2 \rangle \langle \sigma_3 \rangle)^{1-2\nu}} \right)^{\frac{1}{2}} \lesssim \langle \sigma_{res}^{(0)} \rangle^{\frac{1}{q'} - \frac{1}{2} -}$$

with  $\sigma_{res}^{(0)} = \tau + \xi^2 - 2(\xi - \xi_1)(\xi - \xi_2)$  by two applications of Lemma 4.1. Hölder's inequality in  $\xi_1, \xi_2$  leads to

$$t_{0,1} \lesssim \left\| \Sigma_{0,1}(\mu) \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \left( \int \frac{f_1^2(\mu_1)}{\langle \xi_1 \rangle^{1-}} \frac{f_2^2(\mu_2)}{\langle \xi_2 \rangle^{1-}} f_3^2(\mu_3) d\tau_1 d\tau_2 \right)^{\frac{\varrho}{2}} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi}^{r'} L_{\tau}^2}$$

where

$$\Sigma_{0,1}(\mu) := \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{0-} \langle \xi_2 \rangle^{0-} \langle \sigma_{res}^{(0)} \rangle^{-1-} \right)^{\frac{1}{\varrho'}}$$

for  $\varrho = \frac{2q'}{q'+2} +$  and  $\varrho' = \frac{2q'}{q'-2} -$ . The sum  $\Sigma_{0,1}(\mu)$  is uniformly bounded due to Corollary 3.2, estimate (20). Hence,

$$t_{0,1} \lesssim \left\| \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \frac{\|f_1(\cdot, \xi_1)\|_{L^2}^{\varrho}}{\langle \xi_1 \rangle^{\frac{\varrho}{2}-}} \frac{\|f_2(\cdot, \xi_2)\|_{L^2}^{\varrho}}{\langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \|f_3(\cdot, \xi_3)\|_{L^2}^{\varrho} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi}^{r'}}$$

by Minkowski's inequality because  $\varrho \leq 2$ . Now, we apply Hölder's inequality to obtain

$$t_{0,1} \lesssim \left\| \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \frac{\|f_1(\cdot, \xi_1)\|_{L^2}^{r'}}{\langle \xi_1 \rangle^{1-\frac{r'}{q'}+}} \frac{\|f_2(\cdot, \xi_2)\|_{L^2}^{r'}}{\langle \xi_2 \rangle^{1-\frac{r'}{q'}+}} \|f_3(\cdot, \xi_3)\|_{L^2}^{r'} \right)^{\frac{1}{r'}} \right\|_{\ell_{\xi}^{r'}} \quad (28)$$

Hölder's inequality shows

$$\left( \sum_{\xi_i \in \mathbb{Z}} \|f_i(\cdot, \xi_i)\|_{L^2}^{r'} \langle \xi_i \rangle^{\frac{r'}{q'} - 1 -} \right)^{\frac{1}{r'}} \lesssim \|f_i\|_{\ell_\xi^{q'} L_\tau^2}, \quad i = 1, 2$$

Hence, Fubini's theorem provides

$$t_{0,1} \lesssim \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

for any  $1 < q \leq r \leq 2$ , as desired.

*Case 0.2:* We consider the contribution

$$t_{0,2} := \left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{f_1(\mu_1)}{\langle \xi - \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2} - \nu}} \frac{f_2(\mu_2)}{\langle \xi - \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2} - \nu}} \frac{f_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2} - \nu}} d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^2}$$

By replacing the weight  $\langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}}$  by  $\langle \xi - \xi_1 \rangle^{-\frac{1}{2}} \langle \xi - \xi_2 \rangle^{-\frac{1}{2}}$  in the expression  $t_{0,1}$ , the same arguments as in the previous case lead to

$$t_{0,2} \lesssim \left\| \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \frac{\|f_1(\cdot, \xi_1)\|_{L^2}^{r'}}{\langle \xi - \xi_1 \rangle^{1 - \frac{r'}{q'} +}} \frac{\|f_2(\cdot, \xi_2)\|_{L^2}^{r'}}{\langle \xi - \xi_2 \rangle^{1 - \frac{r'}{q'} +}} \|f_3(\cdot, \xi_3)\|_{L^2}^{r'} \right)^{\frac{1}{r'}} \right\|_{\ell_\xi^{r'}}$$

instead of (28) where we used Corollary 3.2, estimate (18) to bound the sum

$$\Sigma_{0,2}(\mu) := \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \langle \sigma_{res}^{(0)} \rangle^{-1-} \right)^{\frac{1}{q'}}$$

By the change of variables  $\xi \mapsto \xi - \xi_1 - \xi_2$  we obtain

$$t_{0,2} \lesssim \left( \sum_{\xi, \xi_1, \xi_2 \in \mathbb{Z}} \frac{\|f_1(\cdot, \xi_1)\|_{L^2}^{r'}}{\langle \xi + \xi_1 \rangle^{1 - \frac{r'}{q'} +}} \frac{\|f_2(\cdot, \xi_2)\|_{L^2}^{r'}}{\langle \xi + \xi_2 \rangle^{1 - \frac{r'}{q'} +}} \|f_3(\cdot, \xi)\|_{L^2}^{r'} \right)^{\frac{1}{r'}}$$

Now, we sum first in  $\xi_1, \xi_2$  and use

$$\sup_{\xi \in \mathbb{Z}} \left( \sum_{\xi_i \in \mathbb{Z}} \|f_i(\cdot, \xi_i)\|_{L^2}^{r'} \langle \xi + \xi_i \rangle^{\frac{r'}{q'} - 1 -} \right)^{\frac{1}{r'}} \lesssim \|f_i\|_{\ell_\xi^{q'} L_\tau^2}, \quad i = 1, 2 \quad (29)$$

to obtain

$$t_{0,2} \lesssim \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

similarly as above.

*Case 1.1:* From now on we have to restrict ourselves to  $2 \leq q' < 4$ . We use duality and consider for  $\varphi \in \ell_\xi^r L_\tau^2$  the quantity  $t_{1,1}$  defined by

$$\begin{aligned} & \sum_{\xi \in \mathbb{Z}} \int \frac{\varphi(\mu)}{\langle \sigma_0 \rangle^{\frac{1}{2} - \nu}} \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{f_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}}} \frac{f_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2} - \nu}} \frac{f_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2} - \nu}} d\tau_1 d\tau_2 d\tau \\ &= \sum_{\xi_1 \in \mathbb{Z}} \int f_1(\mu_1) \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{\varphi(\mu)}{\langle \sigma_0 \rangle^{\frac{1}{2} - \nu}} \frac{f_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2} - \nu}} \frac{f_3(\mu_3)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_3 \rangle^{\frac{1}{2} - \nu}} d\tau d\tau_2 d\tau_1 \end{aligned}$$

Cauchy-Schwarz in  $\tau, \tau_2$  and two applications of Lemma 4.1 show

$$t_{1,1} \lesssim \sum_{\xi_1 \in \mathbb{Z}} \int f_1(\mu_1) \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \sigma_{res}^{(1)} \rangle^{\frac{1}{q'} - \frac{1}{2} -} \left( \int \frac{\varphi^2(\mu) f_2^2(\mu_2) f_3^2(\mu_3)}{\langle \xi_1 \rangle \langle \xi_2 \rangle} d\tau d\tau_2 \right)^{\frac{1}{2}} d\tau_1$$

where with  $\sigma_{res}^{(1)} = \tau_1 + \xi_1^2 + 2(\xi - \xi_1)(\xi - \xi_2)$ . Hölder's inequality in  $\xi, \xi_2$  leads to

$$t_{1,1} \lesssim \sum_{\xi_1 \in \mathbb{Z}} \int f_1(\mu_1) \Sigma_{1,1}(\mu_1) \left( \sum_{\xi, \xi_2 \in \mathbb{Z}} \left( \int \frac{\varphi^2(\mu) f_2^2(\mu_2) f_3^2(\mu_3)}{\langle \xi_1 \rangle^{1-} \langle \xi_2 \rangle^{1-}} d\tau d\tau_2 \right)^{\frac{\varrho}{2}} \right)^{\frac{1}{\varrho}} d\tau_1$$

for  $\varrho = \frac{2q'}{q'+2} +$  and  $\varrho' = \frac{2q'}{q'-2} -$ , where

$$\Sigma_{1,1}(\mu_1) := \left( \langle \xi_1 \rangle^{0-} \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_2 \rangle^{0-} \langle \sigma_{res}^{(1)} \rangle^{-1-} \right)^{\frac{1}{\varrho'}}$$

This is bounded by Corollary 3.2, estimate (21). Cauchy-Schwarz in  $\tau_1$  and Minkowski's inequality provide

$$t_{1,1} \lesssim \sum_{\xi_1 \in \mathbb{Z}} \|f_1(\cdot, \xi_1)\|_{L_\tau^2} \left( \sum_{\xi, \xi_2} \frac{\|\varphi(\cdot, \xi)\|_{L_\tau^2}^\varrho \|f_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho \|f_3(\cdot, \xi_3)\|_{L_\tau^2}^\varrho}{\langle \xi_1 \rangle^{\frac{\varrho}{2}-} \langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}} \quad (30)$$

Now, we use Hölder in  $\xi_1$  to obtain

$$t_{1,1} \lesssim \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \left\| \left( \sum_{\xi, \xi_2} \frac{\|f_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho \|f_3(\cdot, \xi_3)\|_{L_\tau^2}^\varrho \|\varphi(\cdot, \xi)\|_{L_\tau^2}^\varrho}{\langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi_1}^{\varrho'}} \quad (31)$$

By the change of variables  $\xi_2 \mapsto -\xi_2, \xi \mapsto \xi$  the second factor equals

$$\left\| \sum_{\xi, \xi_2} \frac{\|\tilde{f}_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho}{\langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \|\varphi(\cdot, \xi)\|_{L_\tau^2}^\varrho \|\tilde{f}_3(\cdot, \xi_1 - \xi_2 - \xi)\|_{L_\tau^2}^\varrho \right\|_{\ell_{\xi_1}^{\frac{\varrho'}{2}}}$$

where  $\tilde{f}_j = f_j(-\cdot, -\cdot)$ ,  $j = 2, 3$ . This convolution is bounded by

$$\left\| \frac{\tilde{f}_2}{\langle \xi_2 \rangle^{\frac{1}{2}-}} \right\|_{\ell_{\xi_2}^\varrho L_\tau^2} \|\varphi\|_{\ell_\xi^{q'} L_\tau^2} \|\tilde{f}_3\|_{\ell_\xi^{r'} L_\tau^2}$$

due to Young's inequality, because

$$2 + \frac{\varrho}{\varrho'} = 1 + \frac{\varrho}{r} + \frac{\varrho}{r'}$$

Another application of Hölder's inequality with respect to  $f_2$  yields

$$t_{1,1} \lesssim \|\varphi\|_{\ell_\xi^{q'} L_\tau^2} \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

for  $4/3 < q \leq r \leq 2$ .

*Case 1.2:* For the contribution  $t_{1,2}$  the same approach as above leads to

$$t_{1,2} \lesssim \sum_{\xi_1 \in \mathbb{Z}} \|f_1(\cdot, \xi_1)\|_{L_\tau^2} \left( \sum_{\xi, \xi_2} \frac{\|\varphi(\cdot, \xi)\|_{L_\tau^2}^\varrho \|f_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho \|f_3(\cdot, \xi_3)\|_{L_\tau^2}^\varrho}{\langle \xi - \xi_1 \rangle^{\frac{\varrho}{2}-} \langle \xi - \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}}$$

instead of (30) by replacing  $\langle \xi_1 \rangle, \langle \xi_2 \rangle$  in  $t_{1,1}$  by  $\langle \xi - \xi_1 \rangle, \langle \xi - \xi_2 \rangle$ , respectively. The only difference is the use of Corollary 3.2, estimate (18) to bound the sum

$$\Sigma_{1,2}(\mu_1) := \left( \sum_{\substack{\xi, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \langle \sigma_{res}^{(1)} \rangle^{-1-} \right)^{\frac{1}{\varrho}}$$

Hölder's inequality in  $\xi_1$  and then in  $\xi, \xi_2$  provides

$$\begin{aligned} t_{1,2} &\lesssim \|f_1\|_{\ell_{\xi}^{q'} L_{\tau}^2} \left\| \left( \sum_{\xi, \xi_2} \frac{\|f_2(\cdot, \xi_2)\|_{L_{\tau}^2}^{\varrho} \|f_3(\cdot, \xi_3)\|_{L_{\tau}^2}^{\varrho} \|\varphi(\cdot, \xi)\|_{L_{\tau}^2}^{\varrho}}{\langle \xi - \xi_1 \rangle^{\frac{\varrho}{2}-} \langle \xi - \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi_1}^q} \\ &\lesssim \|f_1\|_{\ell_{\xi}^{q'} L_{\tau}^2} \left( \sum_{\xi, \xi_1, \xi_2 \in \mathbb{Z}} \frac{\|\varphi(\cdot, \xi)\|_{L_2}^q}{\langle \xi - \xi_1 \rangle^{1-\frac{q}{q'}+}} \frac{\|f_2(\cdot, \xi_2)\|_{L_2}^q}{\langle \xi - \xi_2 \rangle^{1-\frac{q}{q'}+}} \|f_3(\cdot, \xi_3)\|_{L_2}^q \right)^{\frac{1}{q}} \end{aligned}$$

Now, Hölder in  $\xi$  and the change of variables  $\xi_1 \mapsto \xi - \xi_1$  gives

$$\begin{aligned} &\left( \sum_{\xi, \xi_1, \xi_2 \in \mathbb{Z}} \frac{\|\varphi(\cdot, \xi)\|_{L_2}^q}{\langle \xi - \xi_1 \rangle^{1-\frac{q}{q'}+}} \frac{\|f_2(\cdot, \xi_2)\|_{L_2}^q}{\langle \xi - \xi_2 \rangle^{1-\frac{q}{q'}+}} \|f_3(\cdot, \xi_3)\|_{L_2}^q \right)^{\frac{1}{q}} \\ &\lesssim \|\varphi\|_{\ell_{\xi}^r L_{\tau}^2} \left\| \sum_{\xi_2 \in \mathbb{Z}} \frac{\|f_2(\cdot, \xi_2)\|_{L_2}^q}{\langle \xi - \xi_2 \rangle^{1-\frac{q}{q'}+}} \sum_{\xi_1 \in \mathbb{Z}} \frac{\|f_3(\cdot, \xi_1 - \xi_2)\|_{L_2}^q}{\langle \xi_1 \rangle^{1-\frac{q}{q'}+}} \right\|_{\ell_{\xi}^{\frac{r}{r-q}}}^{\frac{1}{q}} \end{aligned}$$

Let us define

$$\psi(\xi_2) = \sum_{\xi_1 \in \mathbb{Z}} \frac{\|f_3(\cdot, \xi_1 - \xi_2)\|_{L_2}^q}{\langle \xi_1 \rangle^{1-\frac{q}{q'}+}}$$

Young's inequality shows

$$\|\psi\|_{\ell^{\frac{r}{r-q}}_{\xi}} \lesssim \|f_3\|_{\ell_{\xi}^{q'} L_{\tau}^2}^q$$

and therefore

$$\left\| \sum_{\xi_2 \in \mathbb{Z}} \frac{\|f_2(\cdot, \xi_2)\|_{L_2}^q}{\langle \xi - \xi_2 \rangle^{1-\frac{q}{q'}+}} \psi(\xi_2) \right\|_{\ell_{\xi}^{\frac{r}{r-q}}}^{\frac{1}{q}} \lesssim \|f_3\|_{\ell_{\xi}^{q'} L_{\tau}^2} \left\| \sum_{\xi_2 \in \mathbb{Z}} \frac{\|f_2(\cdot, \xi_2)\|_{L_2}^r}{\langle \xi - \xi_2 \rangle^{\frac{r}{q}-\frac{q}{q'}+}} \right\|_{\ell_{\xi}^{\frac{q}{r-q}}}^{\frac{1}{r}}$$

by Hölder's inequality in  $\xi_2$ . Due to the fact that

$$1 + \frac{r-q}{q} = \frac{r}{q'} + \frac{r}{q} - \frac{r}{q'}$$

Young's inequality shows that

$$t_{1,2} \lesssim \|\varphi\|_{\ell_{\xi}^r L_{\tau}^2} \|f_1\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_2\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_3\|_{\ell_{\xi}^{r'} L_{\tau}^2}$$

for all  $4/3 < q \leq r \leq 2$ .

*Case 2.1:* To control the contribution from  $m_{2,1,\nu}$  we exchange the roles of  $f_1$  and  $f_2$  and the arguments from Case 1.1 apply.

*Case 2.2:* To control the contribution from  $m_{2,2,\nu}$  we exchange the roles of  $f_1$  and



$f_2$  and the arguments from Case 1.2 apply.

*Case 3.1:* Fix  $2 \leq q' < 4$  and  $0 \leq \nu < \frac{1}{3q'}$ . By the change of variables

$$\mu_1 \mapsto -\mu_1, \mu_2 \mapsto -\mu_2, \mu \mapsto \mu - \mu_1 - \mu_2 \quad (32)$$

we obtain for the contribution  $t_{3,1}$  the identity

$$\begin{aligned} & \sum_{\xi \in \mathbb{Z}} \int \frac{\varphi(\mu)}{\langle \sigma_0 \rangle^{\frac{1}{2}-\nu}} \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \int \frac{f_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}-\nu}} \frac{f_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}-\nu}} f_3(\mu_3) d\tau_1 d\tau_2 d\tau \\ &= \sum_{\xi \in \mathbb{Z}} \int f_3(\mu) \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \int \frac{\tilde{f}_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \tilde{\sigma}_1 \rangle^{\frac{1}{2}-\nu}} \frac{\tilde{f}_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \tilde{\sigma}_2 \rangle^{\frac{1}{2}-\nu}} \frac{\varphi(\mu_3)}{\langle \tilde{\sigma}_3 \rangle^{\frac{1}{2}-\nu}} d\tau_1 d\tau_2 d\tau \end{aligned}$$

where  $\tilde{\sigma}_1 = \tau_1 - \xi_1^2$ ,  $\tilde{\sigma}_2 = \tau_2 - \xi_2^2$ ,  $\tilde{\sigma}_3 = \tau - \tau_1 - \tau_2 + (\xi - \xi_1 - \xi_2)^2$  and  $\tilde{f}_j = f_j(-\cdot, -\cdot)$ ,  $j = 1, 2$ . Using Cauchy-Schwarz in  $\tau_1, \tau_2$  and Lemma 4.1, the quantity  $t_{3,1}$  is bounded by

$$\sum_{\xi \in \mathbb{Z}} \int f_3(\mu) \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \sigma_{res}^{(3)} \rangle^{\frac{1}{q'} - \frac{1}{2} -} \left( \int \int \frac{\tilde{f}_1^2(\mu_1) \tilde{f}_2^2(\mu_2) \varphi^2(\mu_3)}{\langle \xi_1 \rangle \langle \xi_2 \rangle} d\tau_1 d\tau_2 \right)^{\frac{1}{2}} d\tau$$

where  $\sigma_{res}^{(3)} = \tau - \xi^2 + 2(\xi - \xi_1)(\xi - \xi_2)$ . Hölder's inequality leads to

$$\sum_{\xi \in \mathbb{Z}} \int f_3(\mu) \Sigma_{3,1}(\mu) \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \left( \int \int \frac{\tilde{f}_1^2(\mu_1) \tilde{f}_2^2(\mu_2) \varphi^2(\mu_3)}{\langle \xi_1 \rangle^{1-} \langle \xi_2 \rangle^{1-}} d\tau_1 d\tau_2 \right)^{\frac{\varrho}{2}} \right)^{\frac{1}{\varrho}} d\tau$$

as an upper bound for  $t_{3,1}$  with

$$\Sigma_{3,1}(\mu) = \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{0-} \langle \xi_2 \rangle^{0-} \langle \sigma_{res}^{(3)} \rangle^{-1-} \right)^{\frac{1}{\varrho}}$$

which is uniformly bounded by Corollary 3.2, estimate (20). By Cauchy-Schwarz in  $\tau$  and Minkowski's inequality  $t_{3,1}$  is dominated by

$$\sum_{\xi \in \mathbb{Z}} \|f_3(\cdot, \xi)\|_{L_\tau^2} \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \frac{\|\tilde{f}_1(\cdot, \xi_1)\|_{L_\tau^2}^\varrho \|\tilde{f}_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho \|\varphi(\cdot, \xi_3)\|_{L_\tau^2}^\varrho}{\langle \xi_1 \rangle^{\frac{\varrho}{2}-} \langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}}$$

Now, we recall that  $r \geq \varrho$  for all  $4/3 < q \leq r \leq 2$  and apply Hölder's inequality and Fubini to obtain

$$\begin{aligned} t_{3,1} &\lesssim \|f_3\|_{\ell_\xi^{r'} L_\tau^2} \left\| \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \frac{\|\tilde{f}_1(\cdot, \xi_1)\|_{L_\tau^2}^\varrho \|\tilde{f}_2(\cdot, \xi_2)\|_{L_\tau^2}^\varrho \|\varphi(\cdot, \xi_3)\|_{L_\tau^2}^\varrho}{\langle \xi_1 \rangle^{\frac{\varrho}{2}-} \langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \right)^{\frac{1}{\varrho}} \right\|_{\ell_\xi^r} \\ &\lesssim \|f_3\|_{\ell_\xi^{r'} L_\tau^2} \left( \sum_{\substack{\xi, \xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \frac{\|\tilde{f}_1(\cdot, \xi_1)\|_{L_\tau^2}^r \|\tilde{f}_2(\cdot, \xi_2)\|_{L_\tau^2}^r \|\varphi(\cdot, \xi_3)\|_{L_\tau^2}^r}{\langle \xi_1 \rangle^{1-\frac{r}{q'}+} \langle \xi_2 \rangle^{1-\frac{r}{q'}+}} \right)^{\frac{1}{r}} \quad (33) \end{aligned}$$

Again, Hölder's inequality shows

$$\left( \sum_{\xi_i \in \mathbb{Z}} \|\tilde{f}_i(\cdot, \xi_i)\|_{L^2}^r \langle \xi_i \rangle^{\frac{r}{q'} - 1 -} \right)^{\frac{1}{r}} \lesssim \|f_i\|_{\ell_{\xi}^{q'} L_{\tau}^2}, \quad i = 1, 2$$

Hence,

$$t_{3,1} \lesssim \|f_1\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_2\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_3\|_{\ell_{\xi}^{r'} L_{\tau}^2} \|\varphi\|_{\ell_{\xi}^r L_{\tau}^2}$$

for  $4/3 < q \leq r \leq 2$ , as desired.

*Case 3.2:* To obtain the contribution  $t_{3,2}$  we replace  $\langle \xi_1 \rangle$  and  $\langle \xi_2 \rangle$  in  $t_{3,1}$  by  $\langle \xi - \xi_1 \rangle$  and  $\langle \xi - \xi_2 \rangle$ , respectively. The change of variables (32) transforms  $\langle \xi - \xi_1 \rangle$  to  $\langle \xi - \xi_2 \rangle$  and vice versa and we follow the arguments above to obtain

$$t_{3,2} \lesssim \|f_3\|_{\ell_{\xi}^{r'} L_{\tau}^2} \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \frac{\|\tilde{f}_1(\cdot, \xi_1)\|_{L_{\tau}^2}^r \|\tilde{f}_2(\cdot, \xi_2)\|_{L_{\tau}^2}^r \|\varphi(\cdot, \xi_3)\|_{L_{\tau}^2}^r}{\langle \xi - \xi_1 \rangle^{1 - \frac{r}{q'} +} \langle \xi - \xi_2 \rangle^{1 - \frac{r}{q'} +}} \right)^{\frac{1}{r}}$$

instead of (33), with the only exception that

$$\Sigma_{3,2}(\mu) := \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \langle \sigma_{res}^{(3)} \rangle^{-1-} \right)^{\frac{1}{\theta'}}$$

is controlled by Corollary 3.2, estimate (18). Similar to the Case 0.2, the change of variables  $\xi \mapsto \xi - \xi_1 - \xi_2$  and Hölder's inequality

$$\sup_{\xi \in \mathbb{Z}} \left( \sum_{\xi_i \in \mathbb{Z}} \|f_i(\cdot, \xi_i)\|_{L^2}^r \langle \xi + \xi_i \rangle^{\frac{r}{q'} - 1 -} \right)^{\frac{1}{r}} \lesssim \|f_i\|_{\ell_{\xi}^{q'} L_{\tau}^2}, \quad i = 1, 2$$

yield

$$t_{3,2} \lesssim \|f_1\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_2\|_{\ell_{\xi}^{q'} L_{\tau}^2} \|f_3\|_{\ell_{\xi}^{r'} L_{\tau}^2} \|\varphi\|_{\ell_{\xi}^r L_{\tau}^2}$$

and the estimate for  $T^*$  is done. Finally, we consider the harmless contribution from  $T^{**}$  and show the much stronger estimate

$$\|T^{**}(u_1, u_2, u_3)\|_{X_{r,2}^{\frac{1}{2},0}} \lesssim \|u_1\|_{X_{1,2}^{\frac{1}{2},\frac{1}{3}+}} \|u_2\|_{X_{1,2}^{\frac{1}{2},\frac{1}{3}+}} \|u_3\|_{X_{r,2}^{\frac{1}{2},\frac{1}{3}+}} \quad (34)$$

which immediately yields the desired estimate by trivial embeddings and (27). Indeed, Young's and Hölder's inequality provide

$$\begin{aligned} & \left\| \langle \xi \rangle^{\frac{1}{2}} \xi \int \frac{f_1(\tau_1, \xi)}{\langle \xi \rangle^{\frac{1}{2}} \langle \tau_1 + \xi^2 \rangle^{\frac{1}{3}+}} \frac{f_2(\tau_2, \xi)}{\langle \xi \rangle^{\frac{1}{2}} \langle \tau_2 + \xi^2 \rangle^{\frac{1}{3}+}} \frac{f_3(\tau_3, -\xi)}{\langle \xi \rangle^{\frac{1}{2}} \langle \tau_3 - \xi^2 \rangle^{\frac{1}{3}+}} d\tau_1 d\tau_2 \right\|_{\ell_{\xi}^{r'} L_{\tau}^2} \\ & \lesssim \left\| \left\| \frac{f_1(\tau_1, \xi)}{\langle \tau_1 + \xi^2 \rangle^{\frac{1}{3}+}} \right\|_{L_{\tau_1}^{\frac{6}{5}}} \left\| \frac{f_2(\tau_2, \xi)}{\langle \tau_2 + \xi^2 \rangle^{\frac{1}{3}+}} \right\|_{L_{\tau_2}^{\frac{6}{5}}} \left\| \frac{f_3(\tau_3, -\xi)}{\langle \tau_3 - \xi^2 \rangle^{\frac{1}{3}+}} \right\|_{L_{\tau_3}^{\frac{6}{5}}} \right\|_{\ell_{\xi}^{r'}} \\ & \lesssim \|f_1\|_{\ell_{\xi}^{\infty} L_{\tau}^2} \|f_2\|_{\ell_{\xi}^{\infty} L_{\tau}^2} \|f_3\|_{\ell_{\xi}^{r'} L_{\tau}^2} \end{aligned}$$

This concludes the proof of Theorem 2.4 □

*Proof of Theorem 2.5.* Due to the emdedding (6) the estimate for  $T^{**}$  is already covered by (34). With the same notation as above the estimate (10) for the contribution  $T^*$  is equivalent to

$$\left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int n(\mu, \mu_1, \mu_2) f_1(\mu_1) f_2(\mu_2) f_3(\mu_3) d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^1} \quad (35)$$

$$\lesssim T^\delta \|f_1\|_{\ell_\xi^{q'} L_\tau^2} \|f_2\|_{\ell_\xi^{q'} L_\tau^2} \|f_3\|_{\ell_\xi^{r'} L_\tau^2}$$

where  $n = \langle \sigma_0 \rangle^{-\frac{1}{2}} m$ . We decompose  $n_{k,j} = \langle \sigma_0 \rangle^{-\frac{1}{2}} m_{k,j}$  as above. Again, due to the embedding (6) the stronger estimate (26) already proves estimate (35) for  $n$  replaced by  $n_{k,j,\nu}$  with  $k = 1, 2, 3$ ,  $j = 1, 2$ , corresponding to the Cases 1-3 above. Hence, it is enough to consider the case  $k = 0$  where  $\langle \sigma_0 \rangle$  is the maximal modulation.

*Case 0.1:* Let us fix  $1 < q \leq r \leq 2$  and  $0 \leq \delta < \frac{1}{q'}$ . We proceed similarly to the Case 0.1 in the proof of Theorem 2.4:

$$\widetilde{t_{0,1}} := \left\| \langle \sigma_0 \rangle^{-\frac{1}{2}} \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{f_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}-\nu}} \frac{f_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}-\nu}} \frac{f_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2}-\nu}} d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^1}$$

$$\lesssim \left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{g_1(\mu_1)}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}-\nu-}} \frac{g_2(\mu_2)}{\langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}-\nu-}} \frac{g_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2}-\nu-}} d\tau_1 d\tau_2 \right\|_{\ell_\xi^{r'} L_\tau^p}$$

for any  $p = 2-$ , where we define  $g_j = \langle \sigma_j \rangle^{0-} f_j$  such that

$$\|g_j\|_{\ell_\xi^{q'} L_\tau^p} \lesssim \|f_j\|_{\ell_\xi^{q'} L_\tau^2} \quad (36)$$

Now, by Hölder's inequality and two applications of Lemma 4.1 we get

$$\widetilde{t_{0,1}} \lesssim \left\| \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} \langle \sigma_{res}^{(0)} \rangle^{\frac{1}{q'} - \frac{1}{2} -} \left( \int g_1^p g_2^p g_3^p d\tau_1 d\tau_2 \right)^{\frac{1}{p}} \right\|_{\ell_\xi^{r'} L_\tau^p}$$

with  $\sigma_{res}^{(0)} = \tau + \xi^2 - 2(\xi - \xi_1)(\xi - \xi_2)$ . Hölder's inequality in  $\xi_1, \xi_2$  leads to

$$\widetilde{t_{0,1}} \lesssim \left\| \widetilde{\Sigma_{0,1}}(\mu) \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \left( \int \frac{g_1^p(\mu_1)}{\langle \xi_1 \rangle^{1-}} \frac{g_2^p(\mu_2)}{\langle \xi_2 \rangle^{1-}} g_3^p(\mu_3) d\tau_1 d\tau_2 \right)^{\frac{p}{2}} \right)^{\frac{1}{e}} \right\|_{\ell_\xi^{r'} L_\tau^p}$$

where

$$\widetilde{\Sigma_{0,1}}(\mu) := \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{0-} \langle \xi_2 \rangle^{0-} \langle \sigma_{res}^{(0)} \rangle^{-1-} \right)^{\frac{1}{e'}}$$

for  $\varrho = \frac{2q'}{q'+2}+$  and  $\varrho' = \frac{2q'}{q'-2}-$ . The sum  $\widetilde{\Sigma}_{0,1}(\mu)$  is uniformly bounded due to Corollary 3.2, estimate (20). Hence,

$$\widetilde{t}_{0,1} \lesssim \left\| \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \frac{\|g_1(\cdot, \xi_1)\|_{L^p}^{\varrho}}{\langle \xi_1 \rangle^{\frac{\varrho}{2}-}} \frac{\|g_2(\cdot, \xi_2)\|_{L^p}^{\varrho}}{\langle \xi_2 \rangle^{\frac{\varrho}{2}-}} \|g_3(\cdot, \xi_3)\|_{L^p}^{\varrho} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi}^{r'}}$$

by Minkowski's inequality because  $\varrho \leq p$ . Now, we apply Hölder's inequality and Fubini as in Case 0.1 of the proof of Theorem 2.4 and obtain

$$\widetilde{t}_{0,1} \lesssim \|g_1\|_{\ell_{\xi}^{q'} L_{\tau}^p} \|g_2\|_{\ell_{\xi}^{q'} L_{\tau}^p} \|g_3\|_{\ell_{\xi}^{r'} L_{\tau}^p}$$

for any  $1 < q \leq r \leq 2$ . Finally, (36) proves the desired estimate.

*Case 0.2:* We consider  $\widetilde{t}_{0,2}$  defined as

$$\left\| \langle \sigma_0 \rangle^{-\frac{1}{2}} \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \int \frac{f_1(\mu_1)}{\langle \xi - \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}-\nu}} \frac{f_2(\mu_2)}{\langle \xi - \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}-\nu}} \frac{f_3(\mu_3)}{\langle \sigma_3 \rangle^{\frac{1}{2}-\nu}} d\tau_1 d\tau_2 \right\|_{\ell_{\xi}^{q'} L_{\tau}^1}$$

The same arguments as in the previous case lead to

$$\widetilde{t}_{0,2} \lesssim \left\| \left( \sum_{\xi_1, \xi_2 \in \mathbb{Z}} \frac{\|g_1(\cdot, \xi_1)\|_{L^p}^{\varrho}}{\langle \xi - \xi_1 \rangle^{\frac{\varrho}{2}-}} \frac{\|g_2(\cdot, \xi_2)\|_{L^p}^{\varrho}}{\langle \xi - \xi_2 \rangle^{\frac{\varrho}{2}-}} \|g_3(\cdot, \xi_3)\|_{L^p}^{\varrho} \right)^{\frac{1}{\varrho}} \right\|_{\ell_{\xi}^{r'}}$$

where we used Corollary 3.2, estimate (18) to bound the sum

$$\widetilde{\Sigma}_{0,2}(\mu) := \left( \sum_{\substack{\xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1, \xi_2 \neq \xi}} \langle \xi - \xi_1 \rangle^{0-} \langle \xi - \xi_2 \rangle^{0-} \langle \sigma_{res}^{(0)} \rangle^{-1-} \right)^{\frac{1}{\varrho'}}$$

By the change of variables  $\xi \mapsto \xi - \xi_1 - \xi_2$  we obtain

$$\widetilde{t}_{0,2} \lesssim \left( \sum_{\xi, \xi_1, \xi_2 \in \mathbb{Z}} \frac{\|g_1(\cdot, \xi_1)\|_{L^p}^{r'}}{\langle \xi + \xi_1 \rangle^{1-\frac{r'}{q}+}} \frac{\|g_2(\cdot, \xi_2)\|_{L^p}^{r'}}{\langle \xi + \xi_2 \rangle^{1-\frac{r'}{q}+}} \|g_3(\cdot, \xi)\|_{L^p}^{r'} \right)^{\frac{1}{r'}}$$

Now, we sum first in  $\xi_1, \xi_2$  and use the analogue of (29) for  $\|g_i\|_{L_{\tau}^p}$  ( $i = 1, 2$ ) to obtain

$$\widetilde{t}_{0,2} \lesssim \|g_1\|_{\ell_{\xi}^{q'} L_{\tau}^p} \|g_2\|_{\ell_{\xi}^{q'} L_{\tau}^p} \|g_3\|_{\ell_{\xi}^{r'} L_{\tau}^p}$$

and recall the property (36) of  $g_j$ .  $\square$

*Proof of Remark 3.* Assume that the estimate (11) is valid for some  $b \leq 0$ ,  $1 \leq r \leq \frac{4}{3}$  and  $1 \leq p, q \leq \infty$ . Then for all  $f_i \in \ell_{\xi}^{r'} L_{\tau}^{q'}$  ( $1 \leq i \leq 3$ ) and  $f_0 \in \ell_{\xi}^r L_{\tau}^p$  we have

$$\sum_{\substack{\xi, \xi_1, \xi_2 \in \mathbb{Z} \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \int \frac{\langle \xi \rangle^{\frac{1}{2}} f_0(\mu) f_1(\mu_1) f_2(\mu_2) |\xi_3| f_3(\mu_3)}{\langle \sigma_0 \rangle^{-b} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \sigma_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \langle \sigma_2 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}} \langle \sigma_3 \rangle^{\frac{1}{2}}} d\tau d\tau_1 d\tau_2 < \infty \quad (37)$$

We choose

$$\begin{aligned} f_0(0, \tau) &= \chi(\tau) \text{ and } f_0(\xi, \tau) = 0 \text{ for } \xi \neq 0, \\ f_2(1, \tau_2) &= \chi(\tau_2) \text{ and } f_2(\xi_2, \tau_2) = 0 \text{ for } \xi_2 \neq 1, \\ f_1(0, \tau_1) &= 0 \text{ and } f_1(\xi_1, \tau_1) = \frac{\chi(\tau_1 + (\xi_1 + 1)^2)}{\langle \xi_1 \rangle^{\frac{1}{4}} \ln^{\frac{1}{3}}(\langle \xi_1 \rangle)} \text{ for } \xi_1 \neq 0, \\ f_3(0, \tau_3) &= 0 \text{ and } f_3(\xi_3, \tau_3) = \frac{\chi(\tau_3 - \xi_3^2)}{\langle \xi_3 \rangle^{\frac{1}{4}} \ln^{\frac{1}{3}}(\langle \xi_3 \rangle)} \text{ for } \xi_3 \neq 0, \end{aligned}$$

where  $\chi$  denotes the characteristic function of  $[-1, 1]$ . Then  $f_0 \in \ell_\xi^r L_\tau^p$  and  $f_2 \in \ell_\xi^{r'} L_\tau^{q'}$  for all  $1 \leq r, p, q \leq \infty$  and  $f_1, f_3 \in \ell_\xi^{r'} L_\tau^{q'}$  for all  $r' \geq 4$  and  $1 \leq p, q \leq \infty$ . Let  $I(\xi_1)$  be defined as

$$\begin{aligned} & \int \chi(\tau) \chi(\tau_1 + (\xi_1 + 1)^2) \chi(\tau_2) \chi(\tau - \tau_1 - \tau_2 - (\xi_1 + 1)^2) d\tau d\tau_1 d\tau_2 \\ & \gtrsim \int \chi(\tau) \chi(\tau_1 + (\xi_1 + 1)^2) \chi(\tau - \tau_1 - (\xi_1 + 1)^2) d\tau_1 d\tau \\ & \gtrsim \int \chi(\tau_1 + (\xi_1 + 1)^2) d\tau_1 = 2 \end{aligned}$$

Due to  $\langle \sigma_1 \rangle^{\frac{1}{2}} \lesssim \langle \xi_1 \rangle^{\frac{1}{2}}$  the left hand side of (37) becomes

$$\sum_{|\xi_1| \geq 1} \frac{I(\xi_1)}{\langle \xi_1 \rangle \ln^{\frac{2}{3}}(\langle \xi_1 \rangle)} \gtrsim \sum_{|\xi_1| \geq 1} \frac{1}{\langle \xi_1 \rangle \ln^{\frac{2}{3}}(\langle \xi_1 \rangle)} = \infty$$

which contradicts (37).  $\square$

## 5. THE PROOF OF THE QUINTILINEAR ESTIMATE

Before we start with the proof of Theorem 2.6 we show the following trilinear refinement of the  $L^6$  Strichartz type estimate, see [3], Proposition 2.36. The major point is, that for one of the factors the loss of  $\varepsilon$  derivatives can be avoided. In fact, this refinement also follows by carefully using the decomposition arguments and the Galilean transformation in [3], Section 5. However, we decided to present a proof based on the representation  $\|u\|_{L_{x^t}^6}^3 = \|u^2 \bar{u}\|_{L_{x^t}^2}$  which we learned from [21], in combination with the estimates from Section 3. Similar arguments were already used in [8], cf. Proposition 4.6 and its proof.

**Lemma 5.1.** *For  $\frac{1}{3} < b < \frac{1}{2}$  and  $s > 3(\frac{1}{2} - b)$  the estimate*

$$\|u_1 u_2 \bar{u}_3\|_{L_{x^t}^2} \lesssim \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} \|u_3\|_{X^{0,b}} \quad (38)$$

*holds true.*

*Proof.* We rewrite  $u_1 u_2 \bar{u}_3 = C_1(u_1, u_2, u_3) + C_2(u_1, u_2, u_3)$  for

$$\begin{aligned} C_1(\widehat{u_1, u_2, u_3})(\xi) &= (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \widehat{\bar{u}}_3(\xi_3) \\ C_2(\widehat{u_1, u_2, u_3})(\xi) &= (2\pi)^{-1} \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 = \xi \text{ or } \xi_2 = \xi}} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \widehat{\bar{u}}_3(\xi_3) \end{aligned}$$

where we suppressed the  $t$  dependence. By Plancherel's identity we observe that

$$C_2(u_1, u_2, u_3) = u_1 \int_0^{2\pi} u_2 \bar{u}_3 dy + u_2 \int_0^{2\pi} u_1 \bar{u}_3 dy - u_1 * u_2 * \bar{u}_3$$

where  $*$  denotes convolution with respect to normalized Lebesgue measure on  $[0, 2\pi]$ . Clearly,

$$\|C_2(u_1, u_2, u_3)\|_{L_t^2 L_x^2} \lesssim \prod_{1 \leq k \leq 3} \|u_k\|_{L_t^6 L_x^2} \lesssim \prod_{1 \leq k \leq 3} \|u_k\|_{X^{0, \frac{1}{3}}}$$

by Sobolev estimates in the time variable. For the convolution term we also used Young's inequality. So it remains to prove (38) with  $\|u_1 u_2 \bar{u}_3\|_{L_{xt}^2}$  on the left hand side replaced by  $\|C_1(u_1, u_2, u_3)\|_{L_{xt}^2}$ . Now by Cauchy-Schwarz' inequality and Fubini's theorem (cp. the arguments in the previous section) matters reduce to show that

$$\sup_{\xi, \tau} \Sigma(\xi, \tau) < \infty$$

where  $\Sigma(\xi, \tau)$  is defined as

$$\sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \int \langle \tau_1 + \xi_1^2 \rangle^{-2b} \langle \tau_2 + \xi_2^2 \rangle^{-2b} \langle \tau_3 - \xi_3^2 \rangle^{-2b} d\tau_1 d\tau_2.$$

Using Lemma 4.1 twice, we see that

$$\begin{aligned} \Sigma(\xi, \tau) &\lesssim \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \tau + \xi^2 - 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{2-6b} \\ &\lesssim \left( \sum_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \xi_1 \neq \xi, \xi_2 \neq \xi}} \langle \xi_1 \rangle^{0-} \langle \xi_2 \rangle^{0-} \langle \tau + \xi^2 - 2(\xi - \xi_1)(\xi - \xi_2) \rangle^{\frac{2-6b}{1-2s}-} \right)^{(1-2s)+} \end{aligned}$$

by Hölder's inequality. Since by assumption  $\frac{2-6b}{1-2s} < -1$ , a final application of Corollary 3.2, Part 3, completes the proof.  $\square$

In the  $L_{xt}^2$ -norm on the left hand side of (38) we may, of course, replace any single factor by its complex conjugate. Especially we have

$$\|u_1 u_2 \bar{u}_3\|_{L_{xt}^2} \lesssim \|u_1\|_{X^{0,b}} \|u_2\|_{X^{s,b}} \|u_3\|_{X^{s,b}} \quad (39)$$

Fixing  $u_2$  and  $u_3$  and considering the linear operator

$$X^{0,b} \rightarrow L_{xt}^2 : u_1 \mapsto u_1 u_2 \bar{u}_3$$

we obtain by duality the estimate

$$\|v \bar{u}_2 u_3\|_{X^{0,-b}} \lesssim \|v\|_{L_{xt}^2} \|u_2\|_{X^{s,b}} \|u_3\|_{X^{s,b}} \quad (40)$$

Choosing  $v = u_1 \bar{u}_4 u_5$  and applying (39) (and (38), respectively) again, we have shown the following quintilinear estimate:

**Corollary 5.2.** *Set  $i = 1$  or  $i = 4$ . For  $\frac{1}{3} < b_0 < \frac{1}{2}$  and  $s_0 > 3(\frac{1}{2} - b_0)$  the estimate*

$$\|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X^{0,-b_0}} \lesssim \|u_i\|_{X^{0,b_0}} \prod_{\substack{k=1 \\ k \neq i}}^5 \|u_k\|_{X^{s_0,b_0}} \quad (41)$$

is valid.

In order to prove Theorem 2.6 we shall rely on the interpolation properties of our scale of spaces obtained by the complex method.

**Lemma 5.3.** *Let  $s_i, b_i \in \mathbb{R}$ ,  $1 < r_i, p_i < \infty$  for  $i = 1, 2$ . Then,*

$$(X_{r_0, p_0}^{s_0, b_0}, X_{r_1, p_1}^{s_1, b_1})_{[\theta]} = X_{r, p}^{s, b} \quad (0 < \theta < 1)$$

where

$$\begin{aligned} s &= (1 - \theta)s_0 + \theta s_1, \quad b = (1 - \theta)b_0 + \theta b_1, \\ \frac{1}{r} &= \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}. \end{aligned}$$

*Proof.* The map

$$\mathcal{F} \circ e^{-it\partial_x^2} : X_{r, p}^{s, b} \rightarrow \ell_\xi^{r'}(\langle \xi \rangle^s; L_\tau^{p'}(\langle \tau \rangle^b))$$

is an isometric isomorphism. Here, the image space is the space of sequences in  $\ell_\xi^{r'}$  with weight  $\langle \xi \rangle^s$ , taking values in  $L_\tau^{p'}$  with weight  $\langle \tau \rangle^b$  (with the natural norm). Now, arguing as in [1] Theorem 5.6.3 (replacing  $2^k$  by  $k$ ) and using [1] Theorem 5.5.3 the claim follows.  $\square$

*Proof of Theorem 2.6.* By Sobolev-type embeddings and Young's inequality, we see that for  $r_1, q_1 > 1$ ,  $s_1 > \frac{1}{q_1}$ ,  $b_1 > \frac{1}{3}$  and an auxiliary exponent  $p$  with  $b_1 + \frac{1}{2} > \frac{1}{p} > 5(\frac{1}{2} - b_1)$

$$\begin{aligned} \|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X_{r_1, 2}^{0, -b_1}} &\lesssim \|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X_{r_1, p}^{0, 0}} \\ &\leq \|u_1\|_{X_{r_1, 5p}^{0, 0}} \prod_{i=2}^5 \|u_i\|_{X_{\infty, 5p}^{0, 0}} \\ &\lesssim \|u_1\|_{X_{r_1, 2}^{0, b_1}} \prod_{i=2}^5 \|u_i\|_{X_{q_1, 2}^{s_1, b_1}} \end{aligned} \tag{42}$$

Now fix  $\frac{4}{3} < q \leq r \leq 2$  and  $b > \frac{1}{6} + \frac{1}{3q}$ . We will use complex multilinear interpolation ([1], Theorem 4.4.1) between (41) and (42) with interpolation parameter  $\theta = \frac{1}{2}$ . To do so, we choose

$$s_0 = \frac{3}{2} - \frac{2}{q} - \varepsilon, \quad b_0 = \frac{2}{3q} + \varepsilon$$

in the endpoint (41) and

$$\frac{1}{r_1} = \frac{2}{r} - \frac{1}{2}, \quad \frac{1}{q_1} = \frac{2}{q} - \frac{1}{2}, \quad s_1 = \frac{2}{q} - \frac{1}{2} + \varepsilon, \quad b_1 = \frac{1}{3} + \varepsilon$$

in the endpoint (42), where  $\varepsilon := b - \frac{1}{6} - \frac{1}{3q} > 0$  and use Lemma 5.3. As a result,

$$\|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{X_{r, 2}^{0, -b}} \lesssim \|u_1\|_{X_{r, 2}^{0, b}} \prod_{i=2}^5 \|u_i\|_{X_{q, 2}^{\frac{1}{2}, b}}$$

where in the last line of (42) as well as in the last expression we may exchange  $u_1$  and  $u_4$ . Now our first claim (13) follows from  $\langle \xi \rangle \leq \sum_{1 \leq i \leq 5} \langle \xi_i \rangle$ . We use (6) of Lemma 2.2, that is  $X_{r, 2}^{\frac{1}{2}, -\frac{1}{2}+} \subset X_{r, \infty}^{\frac{1}{2}, -1}$ , and apply (27) with  $\nu = \frac{1}{3q'}$  to all the six norms appearing, which gives a factor  $T^{\frac{2}{q'}-}$ . Finally, (15) follows from (14) and the proof of Theorem 2.6 is complete.  $\square$

## 6. THE GAUGE TRANSFORM

In this subsection we study a nonlinear transformation which turned out to be a key ingredient to the well-posedness theory of the DNLS. This type of transformation for the DNLS was already used by Hayashi-Ozawa [14, 13, 15], later by Takaoka [23], in the nonperiodic case and then adapted to the periodic setting by the second author in [16, 17]. Let us define for  $u \in C([-T, T], L^2(\mathbb{T}))$

$$\mathcal{G}_0 u := \exp(-i\mathcal{I}u) u \quad (43)$$

where

$$\mathcal{I}u(t, x) := \int_0^{2\pi} \int_\theta^x |u(t, y)|^2 - \int_0^{2\pi} |u(t, z)|^2 dz dy d\theta \quad (44)$$

is the unique primitive of

$$x \mapsto |u(t, x)|^2 - \int_0^{2\pi} |u(t, z)|^2 dz$$

with vanishing mean value. Before we study the mapping properties of this transformation let us recall the Sobolev multiplication law in our setting.

**Lemma 6.1.** *Let  $1 < r < \infty$ ,  $0 \leq s \leq s_1$  and  $s_1 > \frac{1}{r}$ . Then,*

$$\|u_1 u_2\|_{\widehat{H}_r^s} \lesssim \|u_1\|_{\widehat{H}_r^{s_1}} \|u_2\|_{\widehat{H}_r^s} \quad (45)$$

*In particular, for  $1 < r < \infty$ ,  $s > \frac{1}{r}$  we have*

$$\|u_1 u_2\|_{\widehat{H}_r^s} \lesssim \|u_1\|_{\widehat{H}_r^s} \|u_2\|_{\widehat{H}_r^s} \quad (46)$$

*Proof.* We have  $\langle \xi \rangle^s \lesssim \langle \xi - \xi_1 \rangle^s + \langle \xi_1 \rangle^{s_1} \langle \xi - \xi_1 \rangle^{s-s_1}$  and by Young's inequality

$$\begin{aligned} \|u_1 u_2\|_{\widehat{H}_r^s} &\lesssim \|\widehat{u}_1\|_{\ell^1} \|\langle \xi \rangle^s \widehat{u}_2\|_{\ell^{r'}} + \|\langle \xi \rangle^{s_1} \widehat{u}_1\|_{\ell^{r'}} \|\langle \xi \rangle^{s-s_1} \widehat{u}_2\|_{\ell^1} \\ &\lesssim \|u_1\|_{\widehat{H}_r^{s_1}} \|u_2\|_{\widehat{H}_r^s} \end{aligned}$$

because  $\|\langle \xi \rangle^{-s_1} \widehat{u}_i\|_{\ell^1} \lesssim \|\widehat{u}_i\|_{\ell^{r'}}$ ,  $i = 1, 2$ .  $\square$

**Lemma 6.2.** *Let  $1 < r \leq 2$  and  $s > \frac{1}{r} - \frac{1}{2}$  or  $r = 2$  and  $s \geq 0$ . Then,*

$$\mathcal{G}_0 : C([-T, T], \widehat{H}_r^s(\mathbb{T})) \rightarrow C([-T, T], \widehat{H}_r^s(\mathbb{T}))$$

*is a locally bi-lipschitz homeomorphism with inverse  $\mathcal{G}_0^{-1} u = \exp(i\mathcal{I}u) u$ .*

*Proof.* We will transfer the ideas from the proof of [16], Lemma 2.3 (where the claim is shown in the  $L^2$  setting) to the case  $1 < r < 2$  and  $s > \frac{1}{r} - \frac{1}{2}$ . Obviously, it suffices to prove

$$\begin{aligned} &\|(\exp(\pm i\mathcal{I}f) - \exp(\pm i\mathcal{I}g)) h\|_{\widehat{H}_r^s} \\ &\lesssim \exp(c\|f\|_{\widehat{H}_r^s}^2 + c\|g\|_{\widehat{H}_r^s}^2) \|f - g\|_{\widehat{H}_r^s} \|h\|_{\widehat{H}_r^s} \end{aligned} \quad (47)$$

for smooth  $f, g, h \in \widehat{H}_r^s$ . By the series expansion of the exponential and (45) we infer that the left hand side of (47) is bounded by

$$\|h\|_{\widehat{H}_r^s} \|\mathcal{I}f - \mathcal{I}g\|_{\widehat{H}_r^{s_1}} \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{k=0}^{n-1} \|\mathcal{I}f\|_{\widehat{H}_r^{s_1}}^k \|\mathcal{I}g\|_{\widehat{H}_r^{s_1}}^{n-1-k}$$

for  $s_1 = \max\{\frac{1}{r}, s\}$ . By the definition of  $\mathcal{I}$  it follows

$$\|\mathcal{I}f - \mathcal{I}g\|_{\widehat{H}_r^{s_1}} \lesssim \| |f|^2 - |g|^2 \|_{\widehat{H}_r^{s_1-1}}$$



In the case  $s \leq \frac{1}{r}$  we have

$$\| |f|^2 - |g|^2 \|_{\widehat{H}_r^{s_1-1}} \lesssim (\|\widehat{f}\|_{\ell^{2-}} + \|\widehat{g}\|_{\ell^{2-}}) \|\widehat{f} - \widehat{g}\|_{\ell^{2-}}$$

Because  $s > \frac{1}{r} - \frac{1}{2}$  we find

$$\|\mathcal{I}f - \mathcal{I}g\|_{\widehat{H}_r^{s_1}} \lesssim (\|f\|_{\widehat{H}_r^s} + \|g\|_{\widehat{H}_r^s}) \|f - g\|_{\widehat{H}_r^s}$$

In the case  $s > \frac{1}{r}$  we use (46) to deduce

$$\| |f|^2 - |g|^2 \|_{\widehat{H}_r^{s_1-1}} \lesssim \|f - g\|_{\widehat{H}_r^s} (\|f\|_{\widehat{H}_r^s} + \|g\|_{\widehat{H}_r^s})$$

and the estimate (47) follows.  $\square$

The following lemma is contained in [17] in the  $r = 2$  case.

**Lemma 6.3.** *Let  $1 < r \leq 2$  and  $s > \frac{1}{r} - \frac{1}{2}$  or  $r = 2$  and  $s \geq 0$ . The translation operators*

$$\begin{aligned} \tau^\mp &: C([-T, T], \widehat{H}_r^s(\mathbb{T})) \rightarrow C([-T, T], \widehat{H}_r^s(\mathbb{T})) \\ \tau^\mp u(t, x) &:= u(t, x \mp 2t \int_0^{2\pi} |u(t, y)|^2 dy) \end{aligned}$$

are continuous. However, their restrictions to arbitrarily small balls are not uniformly continuous.

*Proof.* We only sketch the main ideas and refer to [17], Propositions 3.2.1 and 3.2.2 for details in the  $H^s$  case which easily carry over to the  $\widehat{H}_r^s$  setting: Because the embedding  $\widehat{H}_r^s \subset L^2$  is continuous, the continuity statement follows from the continuity of the map

$$\begin{aligned} \tau &: \mathbb{R} \times C([-T, T], \widehat{H}_r^s(\mathbb{T})) \rightarrow C([-T, T], \widehat{H}_r^s(\mathbb{T})) \\ \tau(h, u)(t, x) &:= u(t, x + ht) \end{aligned}$$

If we fix the time variable, the map

$$\mathbb{R} \times \widehat{H}_r^s(\mathbb{T}) \rightarrow \widehat{H}_r^s(\mathbb{T}), \quad (h, f) \mapsto f(\cdot + h)$$

is continuous. This follows from the fact that a translation by a fixed amount is an isometry in  $\widehat{H}_r^s(\mathbb{T})$  combined with  $e^{i\xi h} \rightarrow e^{i\xi h_0}$  for  $h \rightarrow h_0$  and the dominated convergence theorem. Now, because  $[-T, T]$  is compact, we may approximate  $u \in C([-T, T], \widehat{H}_r^s(\mathbb{T}))$  uniformly by a piecewise constant (in time) function and apply the result for  $t$  fixed.

For  $r > 0$ , the sequences of functions

$$u_{n,j}(t, x) = u_{n,j}(x) = rn^{-s} e^{inx} + c_{n,j}, \quad n \in \mathbb{N}, j = 1, 2$$

with  $c_{n,1} = \frac{1}{\sqrt{n}}$  and  $c_{n,2} = 0$  provide a counterexample to the uniform continuity on balls.  $\square$

As in [16, 17] we define  $\mathcal{G} = \mathcal{G}_0 \circ \tau^-$ , i.e.

$$\mathcal{G}u(t, x) = (\mathcal{G}_0 u) \left( t, x - 2t \int_0^{2\pi} |u(t, y)|^2 dy \right) \quad (48)$$

**Lemma 6.4.** *Let  $u, v \in C([-T, T], H^2) \cap C^1((-T, T), L^2)$  such that  $v = \mathcal{G}u$ . Then,  $u$  solves (DNLS) if and only if  $v$  solves*

$$\begin{aligned} i\partial_t v(t) + \partial_x^2 v(t) &= -iT(v)(t) - \frac{1}{2}\mathcal{Q}(v)(t), \quad t \in (-T, T) \\ v(0) &= \mathcal{G}u(0) \end{aligned} \quad (49)$$

where

$$T(v) = v^2 \partial_x \bar{v} - 2i \int_0^{2\pi} \operatorname{Im} v \partial_x \bar{v} dx v \quad (50)$$

and

$$\mathcal{Q}(v) = \left( |v|^4 - \int_0^{2\pi} |v|^4 dx \right) v - 2 \int_0^{2\pi} |v|^2 dx \left( |v|^2 - \int_0^{2\pi} |v|^2 dx \right) v \quad (51)$$

i.e.  $T(v) = T(v, v, v)$  and  $\mathcal{Q}(v) = Q(v, v, v, v, v)$  for  $T$  and  $Q$  defined in (8) and (12), respectively. Moreover, the map

$$\mathcal{G} : C([-T, T], \hat{H}_r^s(\mathbb{T})) \rightarrow C([-T, T], \hat{H}_r^s(\mathbb{T}))$$

is a homeomorphism with inverse  $\mathcal{G}^{-1} = \tau^+ \circ \mathcal{G}_0^{-1}$ . The restrictions of  $\mathcal{G}$  and  $\mathcal{G}^{-1}$  to arbitrarily small balls fail to be uniformly continuous. However,  $\mathcal{G}$  is locally bi-lipschitz on subsets of functions with prescribed  $L^2$  norm.

*Proof.* To see the equivalence of (DNLS) and (49) the calculations for the periodic case may be found in [16], Section 2. The fact that we may represent  $T$  and  $Q$  via convolution operators on the Fourier side where certain frequency interactions are cancelled out was remarked in [17], Remark 3.2.7. The verification of the precise formulas are straightforward, using (suppressing the  $t$  dependence)

$$\begin{aligned} (2\pi)^{-1} \widehat{v} * \widehat{\partial_x \bar{v}}(0) &= i \int_0^{2\pi} \operatorname{Im} v \partial_x \bar{v} dx \\ (2\pi)^{-1} \widehat{v} * \widehat{\bar{v}}(0) &= \int_0^{2\pi} |v|^2 dx \\ (2\pi)^{-2} \widehat{v} * \widehat{\bar{v}} * \widehat{v} * \widehat{\bar{v}}(0) &= \int_0^{2\pi} |v|^4 dx \end{aligned}$$

The mapping properties follow from Lemma 6.2.  $\square$

*Remark 4.* The cancellation of certain frequency interactions due to the term

$$2 \int_0^{2\pi} \operatorname{Im} v \partial_x \bar{v} dx v,$$

which is crucial for our arguments, cp. (8), is an important feature of the gauge transformation. We observe as well that this expression itself is not well-defined in  $\hat{H}_r^{\frac{1}{2}}(\mathbb{T})$  for  $1 < r < 2$ .

## 7. PROOF OF WELL-POSEDNESS

Now, we show that the Cauchy problem (49) is well-posed. Let  $\chi \in C_0^\infty(\mathbb{R})$  be nonnegative and symmetric such that  $\chi(t) = 0$  for  $|t| \geq 2$  and  $\chi(t) = 1$  for  $|t| \leq 1$ . Recall that  $Z_r^s := X_{r,2}^{s,\frac{1}{2}} \cap X_{r,\infty}^{s,0}$ . We have, similar to the  $L^2$  case, the following linear estimates:

**Lemma 7.1.** *Let  $s \in \mathbb{R}$ ,  $1 < r < \infty$ .*

$$\|\chi S u_0\|_{Z_r^s} \lesssim \|u_0\|_{\widehat{H}_r^s} \quad (52)$$

$$\left\| \chi \int_0^t S(t-t') u(t') dt' \right\|_{Z_r^s} \lesssim \|u\|_{X_{r,2}^{s,-\frac{1}{2}}} + \|u\|_{X_{r,\infty}^{s,-1}} \quad (53)$$

*Proof.* We use the approach from [7], Lemma 3.1. Let  $u_0 \in C^\infty(\mathbb{R})$  be periodic. We calculate  $\mathcal{F}(\chi S u_0)(\tau, \xi) = \mathcal{F}_t \chi(\tau + \xi^2) \mathcal{F}_x u_0(\xi)$  and (52) follows because  $\mathcal{F}_t \chi$  is rapidly decreasing. It suffices to consider smooth  $u$  with  $\text{supp}(u) \subset \{(t, x) \mid |t| \leq 2\}$ . We rewrite

$$\chi(t) \int_0^t S(t-t') f(t') dt' = F_1(t) + F_2(t)$$

where

$$F_1(t) = \frac{1}{2} \chi(t) S(t) \int_{\mathbb{R}} \varphi(t') S(-t') u(t') dt'$$

$$F_2(t) = \frac{1}{2} \chi(t) \int_{\mathbb{R}} \varphi(t-t') S(t-t') u(t') dt'$$

and  $\varphi(t') = \chi(t'/10) \text{sign}(t')$ . Concerning  $\varphi$  we have

$$|\mathcal{F}_t \varphi(\tau)| \lesssim \langle \tau \rangle^{-1} \quad (54)$$

Estimate (52) yields

$$\|F_1\|_{Z_r^s} \lesssim \left\| \int_{\mathbb{R}} \varphi(t') S(-t') u(t') dt' \right\|_{\widehat{H}_r^s}$$

Parseval's equality implies

$$\mathcal{F}_x \left( \int_{\mathbb{R}} \varphi(t') S(-t') u(t') dt' \right) (\xi) = \int_{\mathbb{R}} \overline{\mathcal{F}_t \varphi}(\tau + \xi^2) \mathcal{F}_x u(\tau, \xi) d\tau$$

which gives

$$\left\| \int_{\mathbb{R}} \varphi(t') S(-t') u(t') dt' \right\|_{\widehat{H}_r^s} \lesssim \|u\|_{X_{r,\infty}^{s,-1}}$$

by (54). Now, let us consider  $F_2$ . Due to Young's inequality, we may remove the cutoff function  $\chi$  in front of the integral. The Fourier transform of the remainder is given by

$$\mathcal{F} \left( \int_{\mathbb{R}} \varphi(t-t') S(t-t') u(t') dt' \right) (\tau, \xi) = \mathcal{F}_t \varphi(\tau + \xi^2) \mathcal{F}_x u(\tau, \xi)$$

Estimate (54) proves (53).  $\square$

A standard application of the fixed point argument gives

**Theorem 7.2.** *Let  $\frac{4}{3} < q \leq r \leq 2$ . Then, for every*

$$v_0 \in B_R := \{v_0 \in \widehat{H}^{\frac{1}{2},r}(\mathbb{T}) \mid \|v_0\|_{\widehat{H}^{\frac{1}{2},q}} < R\}$$

*and  $T \lesssim R^{-2q'}$  there exists a solution*

$$v \in Z_r^{\frac{1}{2}}(T) \subset C([-T, T], \widehat{H}_r^{\frac{1}{2}}(\mathbb{T}))$$

of the Cauchy problem (49). This solution is unique in the space  $Z_q^{\frac{1}{2}}(T)$  and the map

$$\left(B_R, \|\cdot\|_{\widehat{H}^{\frac{1}{2},r}}\right) \longrightarrow C([-T, T], \widehat{H}^{\frac{1}{2},r}(\mathbb{T})) : v_0 \mapsto v$$

is locally Lipschitz continuous. Moreover, it is real analytic.

*Sketch of proof.* As a consequence of the estimates (53), (9), (10) and (15)

$$\Phi(v)(t) := \int_0^t e^{i(t-t')\partial_x^2} \left(-\frac{1}{2}\mathcal{Q} - iT\right) v(t') dt'$$

extends to a continuous map  $\Phi : Z_r^{\frac{1}{2}}(T) \rightarrow Z_r^{\frac{1}{2}}(T)$  for  $\frac{4}{3} < r \leq 2$  along with the estimate

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{Z_r^{\frac{1}{2}}(T)} &\lesssim T^{\frac{1}{r}-} \left( \|v_1\|_{Z_r^{\frac{1}{2}}(T)} + \|v_2\|_{Z_r^{\frac{1}{2}}(T)} \right)^2 \|v_1 - v_2\|_{Z_r^{\frac{1}{2}}(T)} \\ &\quad + T^{\frac{2}{r}-} \left( \|v_1\|_{Z_r^{\frac{1}{2}}(T)} + \|v_2\|_{Z_r^{\frac{1}{2}}(T)} \right)^4 \|v_1 - v_2\|_{Z_r^{\frac{1}{2}}(T)} \end{aligned} \quad (55)$$

and with (52) we also have

$$\|e^{it\partial_x^2} v_0 + \Phi(v)\|_{Z_r^{\frac{1}{2}}(T)} \lesssim \|v_0\|_{\widehat{H}^{\frac{1}{2},r_1}} + T^{\frac{1}{r}-} \|v\|_{Z_r^{\frac{1}{2}}(T)}^3 + T^{\frac{2}{r}-} \|v\|_{Z_r^{\frac{1}{2}}(T)}^5$$

Hence, for fixed  $v_0$  the operator  $e^{it\partial_x^2} v_0 + \Phi : D \rightarrow D$  is a strict contraction in some closed ball  $D \subset Z_r^{\frac{1}{2}}(T)$  for small enough  $T$ . By the contraction mapping principle we find a fixed point  $v \in Z_r^{\frac{1}{2}}(T)$  which is a solution of (49) for small times. Similarly, the implicit function theorem shows that the map  $v_0 \mapsto v$  is real analytic, hence locally Lipschitz. Uniqueness in  $Z_q^{\frac{1}{2}}(T)$  follows by contradiction: A translation in time reduces matters to uniqueness for an arbitrarily short time interval which follows from the estimate (55) with  $r = q$ . The lower bound on the maximal time of existence is a consequence of the mixed estimate

$$\|v\|_{Z_r^{\frac{1}{2}}(T)} \lesssim \|v(0)\|_{\widehat{H}_r^{\frac{1}{2}}} + T^{\frac{1}{q}-} \|v\|_{Z_q^{\frac{1}{2}}(T)}^2 \|v\|_{Z_r^{\frac{1}{2}}(T)} + T^{\frac{2}{q}-} \|v\|_{Z_q^{\frac{1}{2}}(T)}^4 \|v\|_{Z_r^{\frac{1}{2}}(T)}$$

for solutions  $v$  and an iteration argument.  $\square$

By combining Theorem 7.2 with Lemma 6.4 and some approximation arguments Theorem 1.2 follows (this is carried out in detail in [11] for the nonperiodic case).

*Remark 5.* In fact, our estimates also imply uniqueness of solutions of (49) in a restriction space based on the  $X_{q,2}^{\frac{1}{2},\frac{1}{2}}$  component only. Hence, the optimal uniqueness statement concerning solutions of (DNLS) provided by our methods is the following: Let  $4/3 < q \leq 2$  and  $u_1, u_2 \in C([-T, T], \widehat{H}_q^{\frac{1}{2}}(\mathbb{T}))$  be solutions of (DNLS) with  $u_1(0) = u_2(0)$ . If additionally  $\mathcal{G}u_1, \mathcal{G}u_2 \in X_{q,2}^{\frac{1}{2},\frac{1}{2}}$  which also satisfy the equation (49), then  $u_1 = u_2$ .

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